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Nonlinear evolution of a weakly unstable wave in a free shear flow with a weak parallel magnetic field

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A study is made of the nonlinear spatial evolution of an externally excited instability wave in a mixing layer of nearly perfectly conducting fluid with a large Reynolds number in a weak parallel magnetic field.

It is shown that the evolution pattern bears a resemblance to that of disturbances in a weakly stratified shear flow with the Prandtl number less than unity which was studied in our earlier publication (Shukhman & Churilov 1997): a weak magnetic field, like a weak stratification when $Pr < 1$, has a stabilizing effect on the nonlinear development of disturbances and in the case when the linear growth rate of the wave is not too large leads either to the instability saturation in the viscous critical layer regime or to the establishment of a unsteady nonlinear critical layer regime where the wave amplitude oscillates without exceeding a certain maximum value. In this case the regime of the quasi-steady nonlinear critical layer is not attained evolutionarily. When the linear growth rate is large enough the magnetic field has no dynamical effect on evolution and the quasi-steady nonlinear critical layer regime with the well-known power-law growth of amplitude ($A \propto x^{2/3}$) is eventually attained.

Also, the critical layer structure and the evolution behaviour in the case of a strong difference of dissipation coefficients (i.e. ordinary viscosity and magnetic viscosity) are considered.

1. Introduction

This paper is a step forward in the study of the weakly nonlinear evolution of unstable disturbances in shear flows of a nearly inviscid fluid to ascertain the connection between the character of this evolution and the neutral mode behaviour at a critical level $y = y_c$ where the flow velocity $v_x = u(y)$ coincides with the wave velocity. The motivation for this study, in general, has been already described in our previous papers (Churilov & Shukhman 1995; Shukhman & Churilov 1997); however, for the purpose of a consistent presentation, we briefly reproduce it here.

It is well known that it is the narrow region surrounding a critical level, i.e. a critical layer (CL), where the most intense wave–flow interaction occurs, which determines the wave's weakly nonlinear dynamics. Outside the CL, the problem is virtually a linear and non-dissipative one, and in the case of a small supercriticality the solution behaviour is described in the main approximation by the neutral mode of an inviscid linearized problem. Inside the CL, on the other hand, these weak effects (e.g. the viscosity, nonstationarity and nonlinearity) become important, and their relative importance depends upon which of the three scales – viscous (ℓ_v), unsteady (ℓ_t) and nonlinear (ℓ_N) – is largest. Here

$$\ell_v = v^{1/3}, \quad \ell_t = |A|^{-1}\mathrm{d}|A|/\mathrm{d}s, \quad \ell_N = |A|^p, \tag{1.1}$$

v is inverse Reynolds number, A is the wave's complex amplitude, p is the parameter dependent on the degree of stratification and varying from 1/2 in a homogeneous medium (as, in particular, in this paper) to 2/3 in a medium with a strong stratification, and s is the evolution variable (the time or downstream coordinate). The CL regime is also determined from the dominant scale, and the CL regimes are said to be viscous, unsteady and nonlinear, respectively.

As is evident from (1.1), the nonlinear and the unsteady scales depend on the wave amplitude and on its rate of variation; consequently, they vary with the evolution. The nonlinear evolution problem is usually stated as a problem of the development (spatial or temporal) of the originally very small unstable disturbance, whose evolution following the attainment of the nonlinearity threshold (i.e. a value of the amplitude at which nonlinear terms become of the same order of magnitude as linear terms) is described by a certain form of the nonlinear evolution equation (NEE). Since the form of the NEE is closely related to the CL regime, a change in the CL scale involves also a change of the form of the NEE and hence the law of evolution. The sequence of evolutionary stages is said to be the evolution scenario.

The predominant role of the CL in the process of nonlinear generation of harmonics indicates the existence of a close connection between the evolution scenario and the neutral mode behaviour at a critical level. And such a connection was indeed revealed (Churilov & Shukhman 1992), and it was shown that there exist two principal scenarios for nonlinear evolution: fast and slow, according to which neutral mode, singular or regular, is at a critical level. It will be recalled that the mode is said to be singular if at least one of its essential (i.e. unsplitting) components (for instance, one of the perturbed velocity components, the density, etc.) has the pole or a branching point.

A fundamental difference between the two evolution scenarios, fast and slow, is as follows.

In the case of the singular neutral mode the nonlinearity threshold is low. This means that the nonlinear stage commences as early as when the CL regime is still linear, i.e. viscous or unsteady (the particular kind depends on the degree of supercriticality corresponding to a given disturbance). If the disturbance 'starts' immediately from the unsteady CL region, i.e. if the supercriticality, measured by the linear growth rate γ_L, is reasonably large, $\gamma_L > v^{1/3}$, upon attaining the nonlinearity threshold, the amplitude begins to grow explosively:†

$$|A| \propto f^{-1/2}(s_0 - s)^{-\alpha}, \quad \alpha > 0, \quad f = \text{const.} \tag{1.2}$$

In this case the unsteady scale

$$\ell_t \sim \gamma \sim |Af^{1/2}|^{1/\alpha} \tag{1.3}$$

grows together with the amplitude so fast, however, that it remains all the time larger than $\ell_N(\sim A^p)$ because typically $1/\alpha < p$. This means that no transition to

† The evolution law (1.2) is an asymptotic solution (when $\gamma \gg \gamma_L$) of the integro-differential NEE, whose structure in a model form may be represented as

$$\mathrm{d}A/\mathrm{d}s = \gamma_L A + fA^3/\ell_t^{2\alpha-1}, \quad \ell_t \sim \gamma = |A|^{-1}\mathrm{d}|A|/\mathrm{d}s.$$

the nonlinear CL regime occurs right up to the amplitudes $A = O(1)$, while larger amplitudes are beyond the scope of weakly nonlinear theory.

If, however, a disturbance 'starts' from the viscous CL region ($\gamma_L < \nu^{1/3}$), it is either stabilized in the viscous CL regime (with a stabilizing sign of the nonlinear term in the Landau–Stuart–Watson NEE that holds in this case) or (with a negative sign) passes into an explosive stage with the law of growth $|A| \propto (s_1 - s)^{-1/2}$, while remaining initially still in the viscous CL regime and subsequently switching over to the unsteady CL regime with the law of growth (1.2). Such a 'fast' scenario is followed by the evolution of disturbances with the singular neutral mode in a stratified flow (Churilov & Shukhman 1988) and in compressible flows (Goldstein & Leib 1989; Leib 1991; Shukhman 1991), and three-dimensional disturbances in a homogeneous incompressible flow (Goldstein & Choi 1989; Wu, Lee & Cowley 1993; Wu 1993*a*, *b*; Churilov & Shukhman 1994; Wu & Cowley 1995; Wu, Lieb & Goldstein 1997).

Whenever, however, the neutral mode is regular, the role of the nonlinearity becomes weaker. The nonlinearity in the case of amplitudes corresponding to the viscous and unsteady CL regimes is still non-competitive and comes into play later, only at the transition to the nonlinear CL regime, i.e. when

$$|A| \gtrsim \max(\nu^{2/3}, \gamma_L^2)$$

(for $p = 1/2$). Upon attaining the boundary of the nonlinear CL regime (coincident in this case with the nonlinearity threshold), the exponential growth with the growth rate γ_L is replaced by a slow power-law growth

$$|A| \propto s^{2/3}. \tag{1.4}$$

More precisely, the evolution law (1.4) starts almost immediately following the transition to the nonlinear CL regime ($A \sim \nu^{2/3}$) for disturbances with $\gamma_L < \nu^{1/3}$, or following some relaxation stage at $A \sim \gamma_L^2$ for disturbances with $\gamma_L > \nu^{1/3}$ (Goldstein & Hultgren 1988). 'Slow' scenarios of such a kind are realized for two-dimensional disturbances in a homogeneous mixing layer (Goldstein & Hultgren 1988; Huerre & Scott 1980; Hultgren 1992), for a zonal flow on the β-plane (Churilov & Shukhman 1987), and for a circular mixing layer (Shukhman 1989).

These two types of scenario differ so that how they can be 'matched' needs unravelling. To clarify this situation requires investigating scenarios of the intermediate type. Such scenarios must be realized in situations where the 'weight' of the singularity in the singular mode is weak, i.e. where the factor f that is responsible for the singularity and hence for the explosive growth (1.2) is small. When $f \ll 1$, the explosive growth rate (1.3) is smaller than when $f = O(1)$. Therefore, the nonlinear scale ℓ_N has time to overtake the unsteady scale $\ell_t \sim \gamma$ still at a reasonably small amplitude $|A| \sim f^{1/(2(\alpha p-1))} \ll 1$. This means that at $f \ll 1$ the transition to the nonlinear CL regime occurs within the framework of the validity of theory, as in the case with the regular neutral mode. Hence, when investigating the evolution of disturbances with the weakly singular neutral mode, we have the opportunity to study the changeover of the evolution character as the factor f varies from zero to values of about unity.

A weak singularity usually manifests itself in the fact that the singular component of the neutral mode nearly entirely detaches itself from the regular one and has a minor effect on it. This is precisely the situation in the two cases which we have already taken up: weakly three-dimensional disturbances in a uniform flow (Churilov & Shukhman 1995) and a weakly stratified flow (Shukhman & Churilov 1997, hereafter referred to as I). In the first case the parameter $(k_z/k_x)^2$ characterizing the degree of the wave's three-dimensional character serves as the factor f, and in the second case the Richardson number Ri does so.

Despite the presence of a number of similar elements in the scenarios in both cases considered above, there are also substantially differing points caused by a difference in the interaction of the regular and singular components of a disturbance. This difference implies that in the case of weakly three-dimensional disturbances such an interaction is driven out to the periphery of the CL and is proceeding in so-called outer diffusion layers, while in a weakly stratified flow such an interaction encompasses the entire CL thickness. These differences lead to different types of NEE and to different types of transitions from one CL regime to another. In particular, the evolution in the regime of *unsteady* nonlinear CL is a new element of the scenario that was first discovered when investigating an instability in a stratified flow and is absent in the problem of weakly three-dimensional disturbances. It is such a regime of evolution where the vorticity inside the CL does not adjust itself to a local instantaneous value of the wave amplitude, and unsteady effects inside the nonlinear CL are more important than dissipative ones. Such a regime of a nonlinear CL differs greatly from a conventional regime of a quasi-steady nonlinear CL invoked in all earlier work, beginning with the pioneering publications of Benney & Bergeron (1969) and Davis (1969) where the role of unsteady terms is negligibly small not only compared with nonlinear but also with dissipative terms. The stage of a nonlinear unsteady CL can set in following the stage of explosive growth in the regime of an unsteady (linear) CL after the nonlinear scale ℓ_N has overtaken the unsteady scale ℓ_t. However, whether such a stage sets in or does not, depends on the interaction character of the regular and singular components and on other details (such as the symmetry properties of the regular component).

To appreciate the extent to which the appearance of the regime of an unsteady nonlinear CL following the stage of explosive growth is typical, it is necessary also to consider other types of flows with the weakly singular neutral mode.

In this paper we consider a homogeneous incompressible flow of nearly perfect conducting fluid with a weak parallel uniform magnetic field. Broadly speaking, in such a flow the neutral mode has two critical levels, and on each of them it is singular. If, however, the magnetic field is sufficiently small, critical levels are close, and under certain conditions a unified common CL can be considered. In this case we have exactly the same formal situation as in the two above-mentioned cases when the singular component (in this case the z-component of the magnetic vector-potential) is detached from the regular component (vorticity) and weakly interacts with it. The square of the Alfvén-to-shear velocity ratio serves here as the factor f governing the 'weight' of the singularity.

It will be shown that the evolution scenario for such a flow very much resembles the scenario which is realized in a weakly stratified flow with less-than-unity Prandtl number, despite the differences in the character of the action of the singular component on the regular component and also in the structure of these two types of flows. As in the case with a weakly stratified flow, a distinguishing feature of this scenario is the existence (when supercriticality is small enough) of the stage of an unsteady nonlinear CL following the stage of explosive growth.

Furthermore, in this paper attention is also centred on two points which were left in the background in earlier work. First, a numerical assessment is made of the boundary in the parameter $\gamma_L/\nu^{1/3}$ separating two types of evolution behaviour in linear (i.e. viscous and unsteady) CL regimes: quasi-stationary development (ending in the instability saturation) and explosive growth (either culminating in the transition to the nonlinear CL regime or falling outside the scope of weakly nonlinear theory). Secondly, a study is made of the CL structure and the disturbance dynamics in the case when two dissipative coefficients involved in the problem differ greatly from each other, which leads to the appearance of an additional diffusion scale.

The paper is organized as follows. In § 2 we report some results of linear analysis to illustrate the overall picture of shear flow stability in a uniform parallel magnetic field and the behaviour of the neutral modes. Section 3 gives the formulation of the problem of the nonlinear evolution, the scaling and the final system of equations whose solutions in the limiting cases of linear (viscous and unsteady) CL regimes are found in § 4, and in the regime of quasi-steady nonlinear CL in § 5. An overall picture of disturbance evolution is constructed in § 6 on the basis of synthesizing results on the evolution in different CL regimes. This Section also gives results derived by investigating the evolution with strongly differing dissipation coefficients. Section 7 discusses the results obtained. Appendices A and B give the derivation of the asymptotic behaviour of the function $\Phi_5(\lambda, P_m)$ introduced for describing the quasi-steady CL regimes, depending on the Haberman parameter λ (Haberman 1972) and the magnetic Prandtl number P_m.

2. Some information from linear theory

The linear stability of magnetofluid shear flows is a subject of numerous papers (see, e.g. Chen & Morrison 1991 and works cited there). We now present some results of linear non-dissipative analysis, but only those aspects which are important for the weakly nonlinear theory.

Consider a shear flow with a monotonic velocity profile $v_x = u(y)$ containing an inflection point along a uniform magnetic field $\boldsymbol{H}_0 = (H_0, 0, 0)$. Two-dimensional disturbances, to which we confine our treatment here, may be represented by the stream function $\psi(x, y)$ and the z-component of the vector-potential of the magnetic field χ, so that $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$; $H_x = \partial\chi/\partial y$, $H_y = -\partial\chi/\partial x$. For disturbances of the form $f(x, y) = \hat{f}(y)\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$ we obtain a linearized equation (see e.g. Kent 1968)

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[(u-c)^2 - c_A^2\right]\frac{\mathrm{d}\hat{\chi}}{\mathrm{d}y} - k^2\left[(u-c)^2 - c_A^2\right]\hat{\chi} = 0, \quad c = \omega/k, \tag{2.1}$$

with the boundary condition $\hat{\chi} \to 0$ when $y \to \pm\infty$. A perturbation of the stream function $\hat{\psi}(y)$ is related to $\hat{\chi}(y)$ by the relation

$$\hat{\psi} = \frac{u-c}{H_0}\hat{\chi} \tag{2.2}$$

and satisfies the equation

$$(u-c)\hat{\Delta}\hat{\psi} - u''\hat{\psi} = c_A^2\hat{\Delta}\left(\hat{\psi}/(u-c)\right), \tag{2.3}$$

which differs from the Rayleigh equation by the presence of a term with c_A on the right-hand side. Here $\hat{\Delta} = \mathrm{d}^2/\mathrm{d}y^2 - k^2$, $c_A = (H_0^2/4\pi\rho)^{1/2}$ is the Alfvén velocity.

Generally speaking, the presence of a parallel uniform magnetic field has a stabilizing effect on an ideal instability of a free monotonic flow profile with an inflection point. To visualize this, we turn to a generalization of Howard's semicircle theorem (e.g. Chandra 1973) which states that complex eigenvalues $c = c_r + \mathrm{i}c_i$ can lie only inside the upper semicircle

$$(c_r - \overline{u})^2 + c_i^2 \leqslant w^2 - c_A^2, \tag{2.4}$$

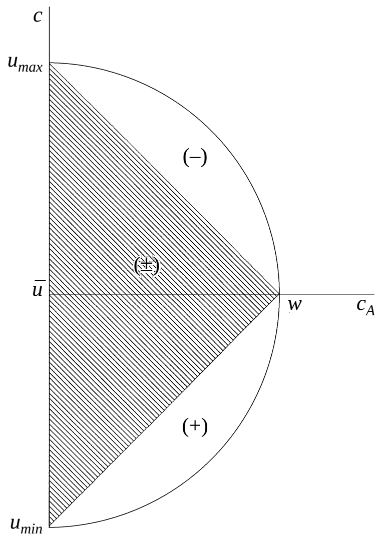

FIGURE 1. The range of possible values of the phase velocity c of the neutral mode as a function of the magnitude of the magnetic field (bounded by a semicircle). The semicircle is divided into three regions in which, respectively, only one resonance (+), only one resonance (−) and both resonances (±) are possible.

where $\bar{u}=\frac{1}{2}(u_{\max}+u_{\min})$, $w=\frac{1}{2}(u_{\max}-u_{\min})$, $u_{\max}=\max\{u(y)\}$, $u_{\min}=\min\{u(y)\}$. It is evident from (2.4) that when $c_A > w$ the flow becomes stable. One can also see that the phase velocity c_r of the neutral mode ($c_i = 0$) is confined within a narrower range compared with the case without a magnetic field:

$$\bar{u} - (w^2 - c_A^2)^{1/2} \leqslant c_r \leqslant \bar{u} + (w^2 - c_A^2)^{1/2}. \tag{2.5}$$

This range is contracted to a point $c_r = \bar{u}$ when $c_A = w$ (see figure 1).

We now direct our attention to the question of critical levels. It is evident from (2.1) that for the flow with a monotonic velocity profile, the neutral mode (if it exists) can have in principle two critical levels: when $y = y_c^+$ and $y = y_c^-$, where

$$u(y_c^{\pm}) = c \pm c_A, \tag{2.6}$$

which merge together when $c_A \to 0$. For a flow without a magnetic field, it follows from (2.5) that $u_{\min} \leqslant c \leqslant u_{\max}$, which means that the neutral mode is certain to have a critical level. In the presence of a field, the question of the existence of a critical level is somewhat more complicated; in this case, however, it is also possible to show that at least one critical level does of necessity exist. Indeed, the resonance (+) is realized if $u_{\min} < c + c_A < u_{\max}$, i.e. if

$$u_{\min} - c_A < c < u_{\max} - c_A, \tag{2.7}$$

and the resonance (−) is realized if $u_{\min} < c - c_A < u_{\max}$, i.e. if

$$u_{\min} + c_A < c < u_{\max} + c_A. \tag{2.8}$$

Regions where the inequalities (2.7) and (2.8) are valid totally cover the entire semicircle on the plane (c, c_A) of possible values of c (figure 1), which does mean the obligatory existence of at least one critical level at any value of c_A smaller than w.

In figure 1 the symbol (+) denotes the region of phase velocities c where only one

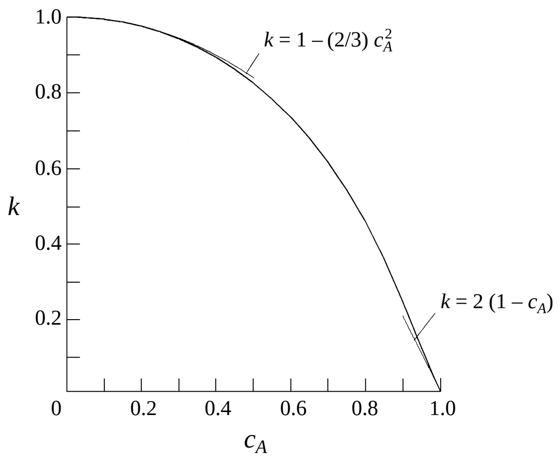

FIGURE 2. Neutral curve for a flow with the velocity profile $u(y) = \overline{u} + \tanh y$ (numerically obtained by the author)

resonance (+) is possible, the symbol (−) corresponds to the region where only one resonance (−) is possible, and both critical levels exist in the region marked by (±).

Note that if the velocity profile is antisymmetric about a mean value, then the neutral mode has a phase velocity $c = \overline{u}$ and is certain to have two critical levels.

Near one of the levels a corresponding Frobenius expansion is

$$\hat{\chi}(y) = a_1\phi_1(y) + a_2\phi_2(y), \quad \hat{\varphi}(y) = (u - c)\hat{\chi}(y)/H_0, \tag{2.9}$$

$$\phi_1(y) = 1+\frac{k^2}{4}\xi^2-\frac{\beta_1}{18}k^2\xi^3+O(\xi^4), \quad \phi_2(y) = \phi_1(y)\ln\xi-\beta_1\xi+\tfrac{1}{2}\left(\beta_1^2-\beta_2-\tfrac{1}{2}k^2\right)\xi^2+O(\xi^3),$$

where $\beta_1 = \left(u_c''/u_c' \pm u_c'/c_A\right)/2$, $\beta_2 = \left(\pm u_c''/c_A + u_c'''/3u_c'\right)/2$, $\xi = y - y_c$. The logarithmic singularity should be understood as the indentation from below (it is assumed that $u_c' > 0$). Note, that here the neutral mode *with two critical levels* is singular (while the neutral mode with one critical level, as it is easy to show, remains regular: $a_2 = 0$).

Since of concern to us in this paper is the case of a weak magnetic field (when the 'weight' of the singularity must also be small), we should give an explicit expression for the neutral mode of some reference flow and study its behaviour with an increase of c_A. Thus, for the neutral mode in a flow model $u(y) = \overline{u} + \tanh y$ we have from (2.1) (selecting the inner narrow region near $y = 0$ with the size $\sim O(c_A)$ and the surrounding external region and using the matched asymptotic expansions method):

$$c = \overline{u}, \quad \omega = k\overline{u}, \quad k \approx 1 - \tfrac{2}{3}c_A^2$$

and

$$\hat{\varphi}(y) \sim \begin{cases} \dfrac{y}{2c_A}\ln\dfrac{y+c_A}{y-c_A}, & |y| \ll 1, \\[2ex] \dfrac{1}{\cosh y}, & |y| \gg c_A, \end{cases} \qquad \hat{\chi}(y) \sim (4\pi\rho)^{1/2} \begin{cases} \dfrac{1}{2}\ln\dfrac{y+c_A}{y-c_A}, & |y| \ll 1, \\[2ex] \dfrac{c_A}{\sinh y}, & |y| \gg c_A. \end{cases} \tag{2.10}$$

The above expressions represent the so-called 'combined' solution to equation (2.1). It it easy to see that (2.10) differs from the familiar solution for a flow without a magnetic field (where $\hat{\varphi} = 1/\cosh y$) only when $y \lesssim c_A \ll 1$.

One can see from (2.10) how the mode is split into the regular ($\hat{\varphi}$) and singular ($\hat{\chi}$) components when $c_A \to 0$ and how critical levels come close together. Note that when $c_A \to 0$ the distance between them is $O(c_A)$, and a correction for the neutral wavenumber k is $O(c_A^2)$ only.

To conclude this Section, for completeness we give the entire neutral curve for this flow on the plane (k, c_A) – see figure 2. The neutral curve on the plane (ω, c_A) is also of the same form because $\omega = k\bar{u} = kc$.

3. Problem statement, scaling and basic equations

We wish to study the influence of a weak magnetic field upon the nonlinear spatial evolution of a weakly unstable wave in a free mixing layer of conducting fluid. The problem statement coincides with that used by Goldstein & Hultgren (1988), and Churilov & Shukhman (1994), and Paper I: the evolution downstream of the flow of a weakly unstable disturbance is considered, which is produced at $x \to -\infty$ by an external source with the frequency ω, somewhat smaller than ω_{cr} which is coincident with the neutral mode frequency (for a given magnetic field): $\omega_{cr} - \omega \ll \omega_{cr}$. The magnetic field is supposed to be small and is taken into account as a correction. Therefore, in the first approximation critical levels merge together to form one, and it coincides with the inflection point, $u_c'' \equiv u''(y_c) = 0$. Designating $\tilde{\chi} = \chi(4\pi\rho)^{-1/2}$ and subsequently omitting the tilde, we write the system of initial equations in the approximation of incompressible magnetic hydrodynamics:

$$\frac{\partial}{\partial t}\Delta\psi + u\frac{\partial}{\partial x}\Delta\psi - u''\frac{\partial\psi}{\partial x} - c_A\frac{\partial}{\partial x}\Delta\chi + \{\Delta\psi, \psi\} - \{\Delta\chi, \chi\} = \nu\Delta^2\psi, \tag{3.1}$$

$$\frac{\partial\chi}{\partial t} + u\frac{\partial\chi}{\partial x} - c_A\frac{\partial\psi}{\partial x} + \{\chi, \psi\} = \nu_m\Delta\chi, \tag{3.2}$$

where $\{a, b\} = a_x b_y - a_y b_x$ and ψ and χ are two-dimensional disturbances of the stream function and of the z-component of the vector potential of the magnetic field in the background of a plane-parallel flow and a homogeneous magnetic field:

$$\psi_0 = \int^y \mathrm{d}y\, u(y), \quad \chi_0 = c_A y. \tag{3.3}$$

Along with the usual viscosity ν, magnetic viscosity ν_m is also involved here. Both viscosities are considered small and need to be taken into account inside the CL only. The ratio $P_m = \nu/\nu_m$ is called the magnetic Prandtl number, and for the time being it will be assumed that $P_m = O(1)$. Note also that in (3.1) a term is omitted, which is responsible for the viscous broadening of an unperturbed flow. Because the effects caused by viscous broadening were studied in some earlier work (e.g. Goldstein & Hultgren 1988; Hultgren 1992; Churilov & Shukhman 1994) and they are not considered in this paper.

It is easy to see that the neutral mode does indeed consist of the regular (ψ) and singular (χ) components on a critical level splitting when $c_A = 0$ and being weakly coupled when $c_A \ll 1$. Therefore, the solution outside the CL (i.e. of the outer problem) in the first approximation is

$$\psi = 2\varepsilon B g(y)\cos\theta, \quad \chi = \frac{c_A}{u - c}\psi; \quad \theta = kx - \omega t + \Theta,$$

where $g(y)$ is the eigenfunction of the neutral mode of the Rayleigh equation (i.e. equation (2.3) with zero right-hand side) with the boundary condition $g \to 0$ when

$y \to \pm\infty$,

$$g = 1 + \alpha_1(y - y_c) + \alpha_2(y - y_c)^2 + \cdots, \quad y \to y_c; \quad \alpha_2 = \left(u_c'''/u_c' + k^2\right)/2,$$

Θ is its phase, $k = \omega_{cr}/c$ is the wavenumber, and $\varepsilon \ll 1$ is the small parameter characterizing the disturbance amplitude; the coefficient α_1 is determined by solving the boundary-value problem for $g(y)$.

The following scaling of quantities involved in the problem will be used:

$$\omega = \omega_{cr} + \varepsilon^{1/2}\Omega, \quad \nu = \eta\varepsilon^{3/2}, \quad \nu_m = \eta_m\varepsilon^{3/2}, \tag{3.4}$$

$$c_A = \varepsilon C_A. \tag{3.5}$$

Scaling (3.4) means that in the analysis we preserve the possibility of studying – in terms of a unified approach – the three CL regimes: viscous, unsteady, and nonlinear, i.e. it is assumed that the respective scales

$$\ell_\nu \sim \nu^{1/3}, \quad \ell_t \sim B^{-1}\mathrm{d}B/\mathrm{d}x, \quad \ell_N \sim \varepsilon^{1/2}$$

are of the same order. It may be worth noting that with such a scaling, as was shown for the first time by Goldstein & Hultgren (1988), the effect of mean-flow divergence becomes important only at downstream distance $x \sim x_\nu \sim \gamma_L/\nu = O\left(\varepsilon^{-1}\right)$, while the main nonlinear effects considered below take place at $x \sim x_N \sim \gamma^{-1} = O\left(\varepsilon^{-1/2}\right)$ which is much less than x_ν. Hence the neglect of the viscous broadening of the underlying mean flow in the governing equations (3.1) and (3.2) is justified (although, in principle, it would be taken into account at the next stage of evolution as is done in Goldstein & Hultgren 1988 and Churilov & Shukhman 1994).

Scaling (3.5) means that we wish to confine ourselves to so small a value of the magnetic field (or more exactly, we wish to consider so large values of the nonlinearity, viscosity and unsteadiness) that it is possible to neglect the splitting of the CL. Indeed, as we have shown in § 2, the distance between critical levels $L \sim O(c_A)$ is (i.e. $O(\varepsilon)$ in view of the scaling (3.5)), while the width of each of the CLs, without regard to in which of the three possible regimes they are, is $O(\varepsilon^{1/2})$, i.e. it is much larger than L. This means that the two CLs actually merge into a single common CL†.

It also follows from (3.5) that the magnetic field influence upon the linear part of the problem may be neglected, i.e. we may disregard the shift of the stability boundary $\Delta\omega_{cr}$ and Δk_{cr} due to magnetic field – they are $O(c_A^2) \sim O(\varepsilon^2)$, while the supercriticality is $O(\varepsilon^{1/2})$.

We introduce a long evolution variable ξ and an inner variable Y:

$$\xi = \varepsilon^{1/2}x, \quad y - y_c = \varepsilon^{1/2}Y.$$

Amplitude B and phase Θ are functions of ξ.

The technique for obtaining evolution equations is well known. The solution of the inner problem according to the (not given here) inner asymptotic representation of

† Scaling (3.5) defines minimal magnetic field which is necessary for non-trivial coupling with the flow field. However the scaling $c_A = O(\varepsilon^{1/2})$ is also possible. In this case the CL width is of the same order as the distance between them. In this case the problem becomes substantially more complicated because the system of equations for the inner problem defies all attempts at analytical study. On the other hand, such a scaling would make it possible to study such an evolution scenario where the interaction of two close CLs would occur. Unquestionably such an interaction is of independent interest and will perhaps become the subject of special study.

Note that recently the case of the finite ($c_A = O(1)$, but $c_A < w$) magnetic field also was considered (Shukhman 1998).

the outer solution is constructed in the form of a series in powers of $\varepsilon^{1/2}$:

$$\psi = \varepsilon\left(\Psi^{(1)} + \varepsilon^{1/2}\Psi^{(2)} + \varepsilon\Psi^{(3)} + \ldots\right), \quad \chi = \varepsilon^{3/2}\phi + \ldots.$$

The first two iterations for ψ do not apply to the magnetic field and are trivially matched to the outer solution: $\Psi^{(1)} = 2B\cos\theta$, $\Psi^{(2)} = 2\alpha_1 BY\cos\theta$. The third iteration for ψ and the first for χ give the desired evolution equations. On introducing $\zeta = \Psi^{(3)}_{YY} - 2B\left(u_c'''/u_c' + k^2\right)\cos\theta$, we write

$$\mathscr{L}_\eta\zeta = -2\frac{u_c'''}{u_c'}\left[\left(\Omega - c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)B\sin\theta + c\frac{\mathrm{d}B}{\mathrm{d}\xi}\cos\theta\right] + \phi_Y\frac{\partial\phi_{YY}}{\partial x} - \phi_{YYY}\frac{\partial\phi}{\partial x} + C_A\frac{\partial\phi_{YY}}{\partial x}, \tag{3.6}$$

$$\mathscr{L}_{\eta_m}\phi = -2kC_A B\sin\theta, \tag{3.7}$$

where

$$\mathscr{L}_\mu = c\frac{\partial}{\partial\xi} + \left(u_c'Y - \Omega/k\right)\frac{\partial}{\partial x} + 2kB\sin\theta\frac{\partial}{\partial Y} - \mu\frac{\partial^2}{\partial Y^2}.$$

The conditions for matching the outer and inner solutions are called the modified solvability conditions (MSC) and, at the order considered, have the form

$$\frac{B}{k}\left(I_0\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega I_1\right) = ⨍_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta\cos\theta\rangle, \quad \frac{I_0}{k}\frac{\mathrm{d}B}{\mathrm{d}\xi} = ⨍_{-\infty}^{\infty}\mathrm{d}Y\,\langle\zeta\sin\theta\rangle, \tag{3.8}$$

where

$$I_1 = ⨍_{-\infty}^{\infty}\mathrm{d}y\frac{u''g^2}{(u-c)^2}, \quad I_2 = \int_{-\infty}^{\infty}g^2\mathrm{d}y, \quad I_0 = cI_1 - 2k^2 I_2.$$

Here $⨍$ stands for a Cauchy principal value, while integrals in (3.8) mean

$$⨍_{-\infty}^{\infty}\mathrm{d}Y(\cdots) = \lim_{Z\to\infty}\int_{-Z}^{Z}\mathrm{d}Y(\cdots), \quad \langle\cdots\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta(\cdots).$$

Equations (3.6), (3.7) are the main equations defining the solution inside the CL and its contribution to MSC (3.8). The vorticity ζ is conveniently represented as the sum of the 'magnetic' (ζ_m) and 'non-magnetic' (ζ_h) parts, $\zeta = \zeta_m + \zeta_h$, so that, instead of (3.6), we obtain

$$\mathscr{L}_\eta\zeta_h = -2\frac{u_c'''}{u_c'}\left[\left(\Omega - c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)B\sin\theta + c\frac{\mathrm{d}B}{\mathrm{d}\xi}\cos\theta\right], \tag{3.9}$$

$$\mathscr{L}_\eta\zeta_m = \phi_Y\frac{\partial}{\partial x}\phi_{YY} - \phi_{YYY}\frac{\partial}{\partial x}\phi + C_A\frac{\partial}{\partial x}\phi_{YY}. \tag{3.10}$$

The system of equations (3.7)–(3.10) must be supplemented with boundary conditions when $|Y|\to\infty$ ensuing from the matching to the outer solution:

$$\phi \sim -\frac{2C_A}{u_c'Y}B\cos\theta + \cdots, \quad \zeta_m \sim \left(\frac{2C_A}{u_c'Y^2}\right)^2 B\cos\theta + \cdots,$$

$$\zeta_h \sim \frac{2u_c'''}{ku_c'^2}\left[\left(\Omega - c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right)B\cos\theta - c\frac{\mathrm{d}B}{\mathrm{d}\xi}\sin\theta\right]Y^{-1} + \cdots.$$

It is interesting to compare the system of equations (3.7)–(3.10) with an analogous system of equations describing the nonlinear evolution of disturbances in a weakly stratified flow (Paper I). It is easy to see that equation (3.7) for ϕ is identical to equation (2.9*a*) of I for density P (in this case, however, the coefficient of magnetic viscosity, rather than the diffusion coefficient for density changes, is involved here); MSC (3.8) is identical to MSC (2.10) of I, and the only formal difference between the two problems lies in the difference of the right-hand sides of equation (3.10) of this paper and equation (2.9*b*) of I for the 'magnetic', ζ_m, and 'stratified', ζ_s, components of the vorticity, respectively. Note also that here, unlike I, none of the values of the magnetic Prandtl number is specific to the disturbance dynamics, while the unity value of the usual Prandtl number made the 'stratified' nonlinear term in the NEE vanish.

Generally speaking, the system of equations (3.7)–(3.10) must be solved numerically as done in I. We, however, proceed as follows. First we consider the evolution regimes corresponding to reasonably small amplitudes when the CL is either viscous or unsteady, and determine the amplitude validity range of these regimes. Next, assuming that, with sufficiently large amplitudes, the regime of a quasi-steady nonlinear CL must be established, we obtain the NEE in this regime also, and we determine its validity range. Further, by synthesizing results on the evolution in different CL regimes, an attempt will be made to describe the entire course of evolution. In doing so, our treatment will be based on the analogy with I where such an analytical study was strengthened by the numerical solution of a system of equations similar to (3.7)–(3.10).

4. Evolution in linear (viscous and unsteady) CL regimes

With a chosen scaling, $B \ll 1$ corresponds to these regimes. The solution of equations (3.7), (3.9) and (3.10) in these regimes is constructed in the form of expansions in powers of B, and it is sufficient to limit oneself to the cubic nonlinearity†. The complex amplitude $\hat{A} = B\exp(\mathrm{i}\Theta)$ is convenient in this case. Since the nonlinearity associated with the 'non-magnetic' part of the vorticity, is non-competitive in these regimes, when calculating ζ_h we need only limit ourselves to the linear in B contribution, which is equivalent to the indentation of the point $y = y_c$ from below in the integral I_1. Upon performing standard calculations (e.g. Goldstein & Leib 1989; Wu *et al.* 1993; Churilov & Shukhman 1994), we obtain the desired evolution equation:

$$\frac{i}{k}\tilde{I}_0\frac{\mathrm{d}\hat{A}}{\mathrm{d}\xi} + \frac{\Omega}{k}\tilde{I}_1\hat{A} = -2\pi C_A^2\,\frac{k^8 u_c'^{\,3}}{c^7}\int_0^\infty \mathrm{d}\xi_1 \times\int_0^\infty \mathrm{d}\xi_2\mathscr{R}(\xi_1,\xi_2;P_m,\Lambda)\hat{A}(\xi-\xi_1)\hat{A}(\xi-\xi_1-\xi_2)\overline{\hat{A}}(\xi-2\xi_1-\xi_2)+\cdots\,, \qquad (4.1)$$

where

$$\mathscr{R} = \int_0^{\xi_1}\mathrm{d}\xi_3\,\xi_1^2\xi_3^2\exp\left\{-\frac{\Lambda}{3}\left[2\xi_1^3-\xi_3^3+3\xi_1^2\xi_2+P_m^{-1}\xi_3^3\right]\right\} +\int_0^{\xi_2}\mathrm{d}\xi_3\,\xi_1^3(\xi_1+2\xi_3)\exp\left\{-\frac{\Lambda}{3}\left[\xi_1^3+3\xi_1^2(\xi_2-\xi_3)+P_m^{-1}\left(\xi_3^3+(\xi_1+\xi_3)^3\right)\right]\right\} -\int_0^{\xi_1}\mathrm{d}\xi_3\,\xi_1^2\xi_3^2\exp\left\{-\frac{\Lambda}{3}\left[\xi_3^3+P_m^{-1}\left(2\xi_1^3-\xi_3^3+3\xi_1^2\xi_2\right)\right]\right\}$$

† It is important to point out that although in equations (3.7), (3.9) and (3.10) describing the CL structure the linear terms are much less than nonlinear ones *for the linear CL regimes*, in MSC (3.8) the linear left side and nonlinear right side are of the same order. Hence, in the final equation (4.1) the nonlinear and nonlinear terms have the same order and can be made to balance.

$$+ \int_0^{\xi_1} d\xi_3\, \xi_1^2\xi_3(2\xi_1-\xi_3)\exp\left\{-\frac{\Lambda}{3}\left[\xi_3^3 + P_m^{-1}\left((2\xi_1-\xi_3)^3 - 3\xi_1(\xi_1-\xi_3)^2 + 3\xi_1^2\xi_2\right)\right]\right\}$$

$$+ \int_0^{\xi_1} d\xi_3\, \xi_3(\xi_1+\xi_2)^2(2\xi_1+2\xi_2-\xi_3)$$

$$\times \exp\left\{-\frac{\Lambda}{3}\left[\xi_3^3 + P_m^{-1}\left((\xi_1+\xi_2)^3 + 3(\xi_1-\xi_3)(\xi_1+\xi_2)^2 + (\xi_1+\xi_2-\xi_3)^3\right)\right]\right\}$$

$$+ \int_0^{\xi_1} d\xi_3\, \xi_2(\xi_1-\xi_3)(\xi_1+2\xi_2+3\xi_3)(2\xi_1+\xi_2)$$

$$\times \exp\left\{-\frac{\Lambda}{3}\left[(\xi_1-\xi_3)^3 + P_m^{-1}\left(4\xi_3^3 + \xi_2^3 + 3\xi_2\xi_3(\xi_2+2\xi_3) + (\xi_1+\xi_2+\xi_3)^3\right)\right]\right\}. \quad (4.2)$$

Here $\Lambda = k^2 u_c'^2 \eta$, the dots mean the non-competitive contribution of the usual 'non-magnetic' nonlinearity proportional to $\hat{A}|\hat{A}|^2/\gamma^3$ and the tilde over I_1 means that the integral is evaluated with the indentation of $y = y_c$ from below. Equation (4.1) describes the evolution in both linear CL regimes: viscous and unsteady, because an arbitrary relation between ℓ_v and ℓ_t is possible here. When $\ell_v \sim \ell_t$ and $\ell_t \gg \ell_v$ it can be solved numerically only (although in the latter case, i.e. in the limit of an unsteady CL, it is easy to find analytically the asymptotic solution).

For the numerical solution, equation (4.1) is conveniently brought to a 'universal' form. By extracting the supercriticality-induced correction to the wavenumber, $\hat{A} = \tilde{A}\exp(\mathrm{i}K\xi)$, $K = \mathrm{Re}(\tilde{I}_1/\tilde{I}_0)\Omega$, and returning to the 'physical' variables, $A = \varepsilon\tilde{A}$, $\nu = \varepsilon^{3/2}\eta$, $c_A = \varepsilon C_A$, $x = \varepsilon^{1/2}\xi$, $\gamma_L = \varepsilon^{1/2}\hat{\gamma}_L \equiv -\varepsilon^{1/2}\mathrm{Im}(\tilde{I}_1/\tilde{I}_0)\Omega$, we obtain

$$\frac{dA}{dx} - \gamma_L A = \frac{2\pi \mathrm{i}}{\tilde{I}_0}\frac{k^8 u_c'^3}{c^7} c_A^2 \int_0^\infty dx_1 \times \int_0^\infty dx_2 \mathscr{R}(x_1, x_2; P_m, \Lambda) A(x-x_1) A(x-x_1-x_2) \overline{A}(x-2x_1-x_2). \quad (4.3)$$

As $x \to -\infty$ the disturbance grows exponentially, $A = A_0\exp(\gamma_L x)$. We put $A(x) = A_0 a(x)\exp(\gamma_L x)$, $a(x) \to 1$ when $x \to -\infty$, and introduce a new evolution variable $T = T_0\exp(2\gamma_L x)$, $T_0 = 2\pi|\tilde{I}_0|^{-1}|A_0|^2 c_A^2 (2\gamma_L)^{-8} k^8 u_c'^3 c^{-7}$. In these variables, equation (4.3) assumes the form

$$\frac{da}{dT} = \mathrm{e}^{-\mathrm{i}\varphi}\int_0^\infty ds\, s^6 \mathrm{e}^{-s}\int_0^1 d\sigma\, \frac{\mathscr{K}(\sigma, s; P_m, \Gamma)}{(1+\sigma)^7}\, a\left(T\mathrm{e}^{-s/(1+\sigma)}\right) a\left(T\mathrm{e}^{-\sigma s/(1+\sigma)}\right) \overline{a}\left(T\mathrm{e}^{-s}\right), \quad (4.4)$$

with initial condition $a(T=0)=0$. Here only the phase φ defined by the relationship $\mathrm{i}\tilde{I}_0^{-1} = |\tilde{I}_0|^{-1}\exp(-\mathrm{i}\varphi)$ depends on the flow structure. The explicit form of the kernel $\mathscr{K}$ may be easily obtained from (4.2). Here parameter $\Gamma = 2c\gamma_L/(k^2 u_c'^2 \nu)^{1/3}$ characterizes the relative role of the viscosity and unsteadiness in the CL at the initial stage of evolution when the unsteady scale ℓ_t is defined by the linear growth rate γ_L. When $\Gamma \ll 1$ we are dealing with a disturbance starting in the viscous CL regime, and when $\Gamma \gg 1$ the unsteady CL regime is at work†.

First we consider the limiting cases of viscous and unsteady CLs.

† Of course, upon reaching the nonlinearity threshold, the unsteady scale ℓ_t is defined by the current growth rate $\gamma = |A|^{-1}d|A|/dx$ and the CL regime is no longer defined by the parameter Γ only.

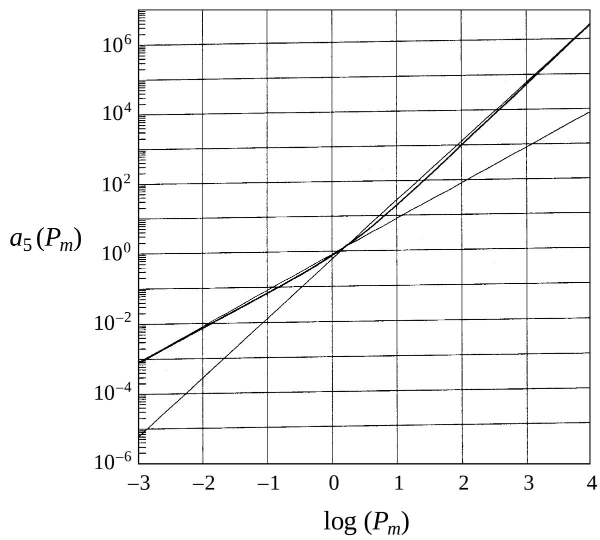

FIGURE 3. Dependence of the function a_5, involved in the Landau constant, on the magnetic Prandtl number P_m (heavy curve). Thin lines show asymptotic expansions (4.6)

4.1. *Viscous CL regime*

To this case there corresponds a formal limit $\Lambda \to \infty$ in (4.3) (or $\Gamma \to 0$ in (4.4)). Because of the fast decreasing exponents, the main contribution to the integral on the right-hand side is made by small delays. The nonlinearity becomes local in x, transforming equation (4.3) to a Landau–Stuart–Watson equation:

$$\frac{\mathrm{d}A}{\mathrm{d}x} - \gamma_L A = \frac{8\mathrm{e}^{-\mathrm{i}\varphi}}{|\tilde{I}_0|} \frac{k^{10/3} u_c'^{-5/3}}{\nu^{7/3}} c_A^2\, a_5(P_m) A|A|^2. \tag{4.5}$$

We will not give here the unwieldy analytical expression for the function $a_5(P_m)$ involved in the Landau constant†. It contains many integrals which have to be computed numerically, however. Therefore, we limit ourselves only to a plot of a_5 (see figure 3) and to asymptotic representations at large and small P_m. We have

$$a_5(P_m) = \begin{cases} a_5^{(l)} P_m^{5/3} + O(P_m^{4/3}), & P_m \gg 1, \\ a_5^{(s)} P_m + O(P_m^{4/3}), & P_m \ll 1, \end{cases} \tag{4.6}$$

$$a_5^{(l)} = (\tfrac{1}{6}\pi)\Gamma^2(2/3)\,(3/4)^{1/3} \left[1 - 2^{5/3} \int_0^1 \frac{(1-z^2)(1+z)\,\mathrm{d}z}{(z^3 + 6z^2 + 3z + 2)^{5/3}}\right] = 0.58154\ldots,$$

$$a_5^{(s)} = |a_2|/2, \quad a_2 = -(\tfrac{1}{6}\pi)\Gamma\,(1/3)\,(3/2)^{1/3} = -1.6057\ldots,$$

(for $P_m = 1$, $a_5 = 0.7257\ldots$). Here $\Gamma(x)$ is the Euler function. The constant a_2 is a coefficient in the asymptotic expansion of the function $\Phi_2(\lambda)$ when $\lambda \gg 1$: $\Phi_2 \approx -\pi + a_2\lambda^{-4/3}$ (Churilov & Shukhman 1996). An important point is that $a_5(P_m)$ is positive for all P_m. Since $\operatorname{Im}\tilde{I}_0 = \pi c u_c'''/u_c'^2 < 0$, the real part of the coefficient of $A|A|^2$ (i.e. the Landau constant) is negative ($\cos\varphi < 0$), and equation (4.5) describes

† The designation is introduced by analogy with I where a similar function $a_4(Pr)$ was also used.

a stabilization on the level $|A| = A_{sat}$:

$$A_{sat} = d_1 \left(\frac{\nu^{7/3}\gamma_L}{c_A^2} \right)^{1/2} \sim A_1, \quad d_1 = \left[\frac{u_c'^{5/3}|\tilde{I}_0|}{8k^{10/3}a_5(P_m)|\cos\varphi|} \right]^{1/2}. \tag{4.7}$$

4.2. *Unsteady CL regime*

Assuming $\Lambda \to 0$ in (4.3) (or $\Gamma \to \infty$ in (4.4)) we obtain from (4.3):

$$\frac{\mathrm{d}A}{\mathrm{d}x} - \gamma_L A = \frac{4\pi\,\mathrm{e}^{-\mathrm{i}\varphi}}{3|\tilde{I}_0|}\frac{k^8 u_c'^3}{c^7}c_A^2 \int_0^\infty \mathrm{d}s\, s^6 \int_0^1 \mathrm{d}\sigma\, \sigma^3(1+\sigma)^2 A(x-s)A(x-\sigma s)\overline{A}\left(x-(1+\sigma)s\right), \tag{4.8}$$

or, in 'universal' variables, from (4.4)

$$\frac{\mathrm{d}a}{\mathrm{d}T} = \tfrac{2}{3}\mathrm{e}^{-\mathrm{i}\varphi} \int_0^\infty \mathrm{d}s\, s^6\, \mathrm{e}^{-s} \int_0^1 \mathrm{d}\sigma\, \frac{\sigma^3(1+\sigma^2)}{(1+\sigma)^7}\, a\left(T\mathrm{e}^{-s/(1+\sigma)}\right) a\left(T\mathrm{e}^{-\sigma s/(1+\sigma)}\right) \overline{a}\left(T\mathrm{e}^{-s}\right). \tag{4.9}$$

The structure of such an equation has been investigated on numerous occasions (e.g. Hickernell 1984; Churilov & Shukhman 1988; Goldstein & Choi 1989). It is easy to describe the character of its solution. The stage of exponential growth with a growth rate γ_L and the attainment of the nonlinearity threshold

$$|A| = A_2 \sim \gamma_L^4/c_A \tag{4.10}$$

is followed by an explosive growth according to the law

$$A \sim c_A^{-1}(x_0 - x)^{-4+\mathrm{i}\beta(\varphi)} \quad \left(\text{or} \quad a \sim (T_0 - T)^{-4+\mathrm{i}\beta(\varphi)}\right). \tag{4.11}$$

This expression represents an exact solution of equation (4.8) with the term $\gamma_L A$ omitted. The behaviour of the argument of the complex amplitude a and the 'time of explosion' T_0 depend on the φ. Because of the ambiguity of the function $\beta(\varphi)$ (e.g. Shukhman 1991), we have no prior knowledge of the particular branch for the solution when $T \to T_0$. The time T_0 also cannot be calculated analytically. Therefore, this equation in the universal form (4.9) was solved numerically for $\varphi = \arctan(\frac{1}{2}\pi) + (\frac{1}{2}\pi) \approx 0.8195\pi$, which corresponds to the hyperbolic tangent profile $u = \overline{u} + \tanh y$ with $\overline{u} = 1$ (for which $\tilde{I}_0 = -2(2 + \mathrm{i}\pi\overline{u})$, $\tilde{I}_1 = -2\mathrm{i}\pi$). The outcome is shown in figure 5 (the pertinent curve is marked as $10 \leqslant \Gamma \leqslant \infty$).

4.3. *The intermediate case*

Despite the 'stabilizing' sign of the nonlinear term in (4.9) ($\cos\varphi < 0$) the disturbance reaches an explosive stage, provided it starts from the region of unsteady CL, $\Gamma \gg 1$, unlike the disturbance starting from the region of viscous CL, $\Gamma \ll 1$. In principle, these two limiting cases suffice to produce a picture of the evolution in linear CL regimes. However, it is also interesting to study the character of the transition from one regime to another as the parameter Γ varies. It is for this purpose only that a numerical solution of NEE (4.4) for arbitrary values of Γ was sought (see also Goldstein & Leib 1989; Leib 1991; Wu & Cowley 1995).

As in §4.2, calculations were carried out for $\varphi \approx 0.8195\pi$ and it was assumed that $P_m = 1$. The evolution of disturbances starting from the region of reasonably small Γ, $\Gamma < 0.8$, is shown in figure 4 on the plane $\left(T, \Gamma^4|a(T)|T^{1/2}\right)$. The quantity plotted here on the ordinate axis, is proportional to the amplitude A: $A = d\Gamma^4 a(T)T^{1/2}$, $d = (|\tilde{I}_0|/2\pi)^{1/2}c_A^{-1}c^{-1/2}k^{-4/3}u_c'^{7/6}\nu^{4/3}$. For such values of Γ, there is not yet any explosive growth. One can see that, with decreasing Γ, the saturation amplitude

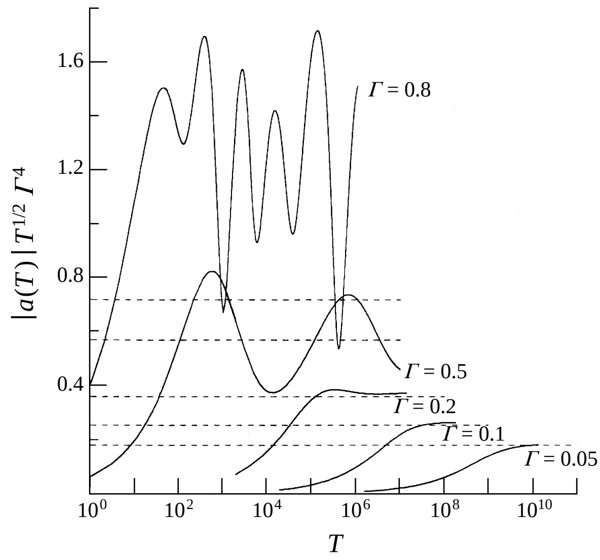

FIGURE 4. Solution of NEE (4.4) describing the evolution in linear CL regimes, at a not too large supercriticality ($\Gamma \equiv 2\gamma_L/\nu^{1/3} \leqslant 0.8$) and with the same dissipation coefficients ($P_m = 1$) and $\varphi = 0.8195\pi$.

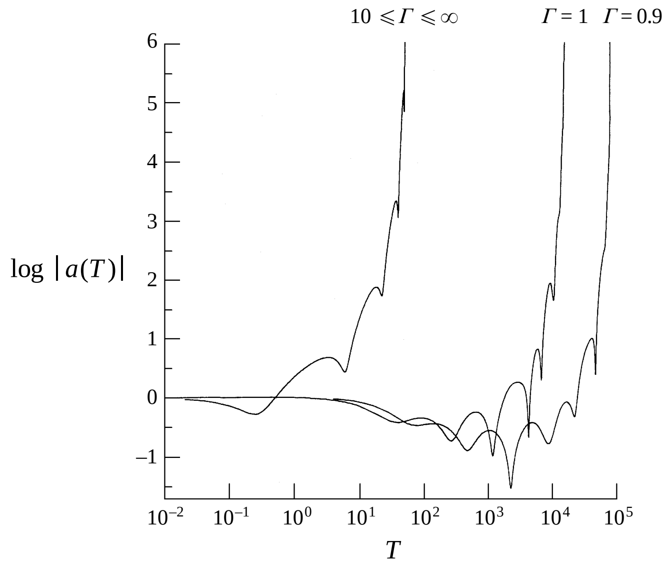

FIGURE 5. Same as figure 4 but for a reasonably large supercriticality ($0.9 \leqslant \Gamma \leqslant \infty$).

comes closer to the value given in (4.7) (dashes) calculated from the approximate equation (4.5).

Figure 5 illustrates the explosive growth of disturbances starting from the region $\Gamma > 0.9$. It is evident that when $\Gamma > 10$ the evolution follows the scenario for a purely unsteady CL ($\Gamma = \infty$). Thus, a critical value of Γ (for the given $P_m = 1$) turns out to be localized in the region $0.8 < \Gamma < 0.9$, which is in reasonably good agreement with elementary estimates.

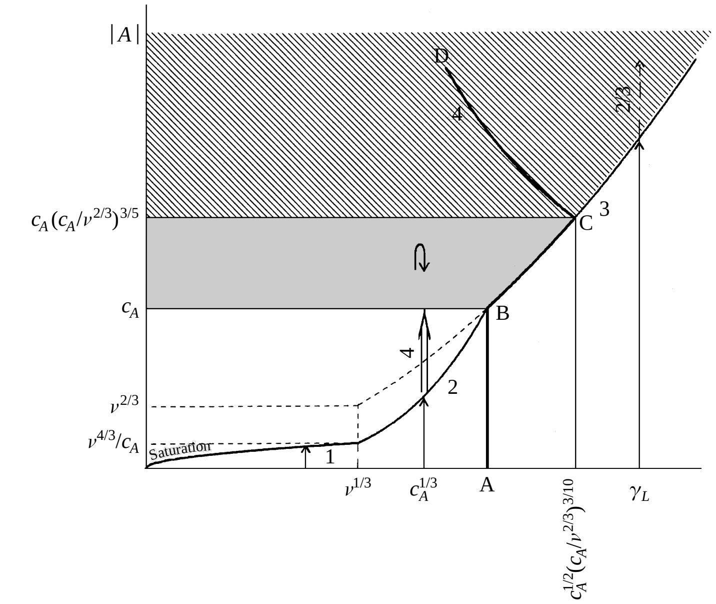

FIGURE 6. Diagram of disturbance development at $P_m = 1$. Case $c_A \gg \nu^{2/3}$. Moderately heavy lines show nonlinearity thresholds; curve 1, $A_1 = (\gamma_L \nu^{7/3}/c_A^2)^{1/2}$; curve 2, $A_2 = \gamma_L^4/c_A$; curve 3, $A_3 = \gamma_L^2$. The arrows indicate the various stages of evolution: $\longrightarrow A \sim \exp(\gamma_L x)$, $\overset{\alpha}{\Longrightarrow} A \sim (x_0 - x)^{-\alpha}$, $- \cdot \overset{\beta}{-} \cdot \rightarrow A \sim \xi^{\beta}$. Curve 4, $A_4 = (c_A^2/\nu^{1/2}\gamma_L)^{4/3}$. The heavy line ABCD bounds from the right the region where the magnetic nonlinearity is dominant. Cross-hatching shows the quasi-steady CL region. The region (gap) of the unsteady nonlinear CL is filled.

4.4. *The resulting picture of evolution in linear CL regimes*

Explosive growth according to the law $|A| \sim c_A^{-1}(x_0 - x)^{-4}$ (when $\gamma_L < c_A^{1/2}$) is not the final stage of evolution: because of the smallness of the magnetic field, the unsteady scale $\ell_t \sim \gamma \sim \left(c_A^2|A|^2\right)^{1/8}$ does not increase sufficiently fast with increasing amplitude, and when

$$|A| \sim c_A \tag{4.12}$$

the nonlinear scale $\ell_N \sim |A|^{1/2}$ overtakes it – there occurs a transition to the regime of nonlinear CL. As in I, this transition is realized from the explosive stage of the unsteady CL.

For disturbances with a sufficiently small supercriticality, $\gamma_L < \nu^{1/3}$, for which the saturation is the final stage of evolution, the nonlinearity threshold $A_1(\gamma_L)$, coinciding in this case in order of magnitude with $A_{sat}(\gamma_L) \sim \left(\nu^{7/3}\gamma_L/c_A^2\right)^{1/2}$, lies below the formal boundary of the transition to the nonlinear CL ($A \sim \nu^{2/3}$) throughout the region of viscous CL ($\gamma_L < \nu^{1/3}$) only if

$$c_A > \nu^{2/3}. \tag{4.13}$$

In this case the evolution from start to finish proceeds in the regime of viscous CL, i.e. equation (4.5) remains valid in the entire course of the evolution.

In the case

$$c_A < \nu^{2/3} \tag{4.14}$$

however, saturation in the viscous CL regime is possible only for disturbances with a sufficiently small γ_L $\left(\gamma_L < c_A^2/\nu < \nu^{1/3}\right)$. Disturbances with a larger γ_L

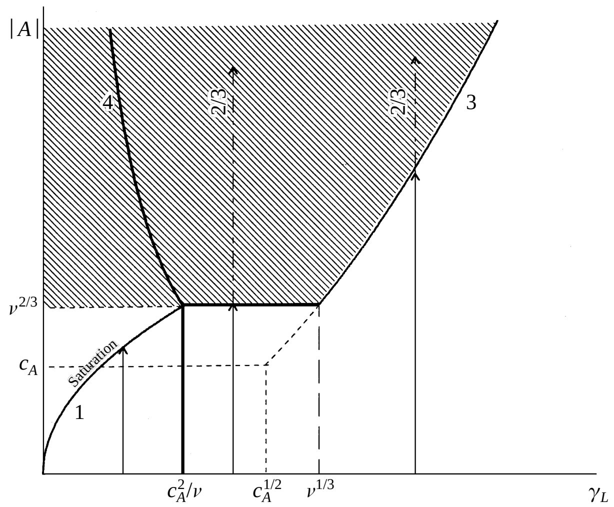

FIGURE 7. Same as in figure 6 but for the case $c_A \ll \nu^{2/3}$.

$\left(c_A^2/\nu < \gamma_L < \nu^{1/3}\right)$, without reaching saturation (or, equivalently, the nonlinearity threshold), find themselves in the nonlinear CL regime.

Therefore, for constructing a full picture, it is necessary to distinguish two cases of relationships between parameters of the medium†: $c_A > \nu^{2/3}$ and $c_A < \nu^{2/3}$.

Figure 6 shows the amplitude–supercriticality diagram for the case $c_A > \nu^{2/3}$. The lower part of the diagram $\left(|A| < \max\{c_A, \gamma_L^2\}\right)$ corresponds to the linear CL regimes under discussion. Different kinds of lines show CL regime boundaries and nonlinearity thresholds, and different arrows show the laws of amplitude variation corresponding to these regimes. In the region $\gamma_L > c_A^{1/2}$ the magnetic nonlinearity is no longer capable of having a substantial influence upon the evolution, and the usual 'non-magnetic' nonlinearity is not yet competitive here. Therefore, this region involves the usual exponential growth of disturbances with the growth rate γ_L right up to the boundary of the transition to the nonlinear CL regime: $|A| \sim A_3 \sim \gamma_L^2$.

Figure 7 presents the case $c_A < \nu^{2/3}$. For the time being, only the part of the diagram where $|A| < \max\{\nu^{2/3}, \gamma_L^2\}$ is discussed. Here the region of values of γ_L where the magnetic field influences the evolution is narrower, $\gamma_L < c_A^2/\nu$. At larger γ_L the evolution is almost the same as at a corresponding value of γ_L without a field: growth according to the law $\sim \exp(\gamma_L x)$ prior to the transition to the nonlinear CL regime at $A \sim \nu^{2/3}$, if $\gamma_L < \nu^{1/3}$, or at $A \sim \gamma_L^2$, if $\gamma_L > \nu^{1/3}$. Note that with such small values of the magnetic field, elements of the 'fast' scenario (see § 1) do not manifest themselves yet: there is no stage of explosive amplitude growth, and the transition to the nonlinear CL (if there is no saturation in the viscous CL regime) proceeds immediately from the linear stage, as in scenarios with the regular neutral mode.

Thus, we have ascertained that at certain values of parameters the evolution does not culminate in saturation in the linear CL regimes, but the transition to the nonlinear CL regime occurs.

To advance further in the study, as has already been pointed out in § 3, it is necessary to have the numerical solution of the 'exact' system of equations (3.7)–(3.10). We, however, shall not seek such a numerical solution, but avail ourselves of

† It will be recalled that in the weakly-stratified problem it was also necessary to distinguish two cases: $Ri < \nu^{2/3}$ (Churilov & Shukhman 1996) and $Ri > \nu^{2/3}$ (Paper I).

the analogy of the problem at hand with the problem of disturbance evolution in a weakly stratified flow considered in I where we carried out a similar numerical investigation. This analogy can be extended to the variant of the problem from I where the nonlinearity due to stratification, has, as in this case, a stabilizing sign, i.e. to the case $Pr < 1$.

For this analogy to be realized, we proceed as follows. Suppose that after the transition to the nonlinear CL regime and some relaxation stage the regime of *quasi-steady nonlinear CL* is established, i.e. an evolution regime where the vorticity and the magnetic field inside the CL can be considered to be instantaneously adjusting to a local instantaneous value of amplitude (as if they were following it). Such a regime is amenable to analytical study, and we can obtain a corresponding evolution equation and determine its validity range. The equation to be obtained for such a regime, as will be shown later in the text, resembles closely the corresponding equation (i.e. equation (4.8*a*)) of I. This will permit us to describe the evolution in the present case, based on the similarity of the equations and on results of a numerical study obtained in I.

5. Regime of quasi-steady nonlinear CL (hypothetical)

The word 'hypothetical' appearing in the heading means that, in fact, we do not yet know if the realization of this regime in the course of the evolution is possible. Indeed, in a weakly stratified flow (I), when $Pr < 1$, such a regime could be attained only at sufficiently large values of γ_L, at which stratification does not play a substantial role. It turns out that in this case we have a similar situation†. Nevertheless, studying this regime and its validity range helps at a qualitative level to understand the character of the solution which could be obtained through a numerical study of the 'exact' system of equations.

Now, we suppose that the amplitude has sufficiently grown, all relaxation processes are over, and the regime of quasi-steady nonlinear CL is attained‡. The quasi-steadiness in the nonlinear CL regime implies the smallness of the evolution term in the operators $\mathscr{L}$ compared not only with the nonlinear but also the viscous term, which corresponds to the inequalities

$$\ell_t \ll \ell_\nu^3/\ell_N^2 \ll \ell_N \quad \text{or} \quad \hat{\gamma} \ll \eta/B \ll B^{1/2}\,, \quad \hat{\gamma} = \left|B^{-1}\mathrm{d}B/\mathrm{d}\xi\right| \equiv \varepsilon^{-1/2}\gamma. \tag{5.1}$$

The meaning of the right-hand inequality (5.1) is obvious. The left-hand inequality means that unsteady processes which are still important in regions of so-called outer diffusion layers (ODL) (see, for example, Churilov & Shukhman 1995), having the scale

$$l_{ODL} \sim (\nu/\gamma)^{1/2} \sim (\ell_\nu^3/\ell_t)^{1/2},$$

are expelled together with these layers to the distant periphery of the CL: $l_{ODL} \gg \ell_N$.

Equations (3.7), (3.9) and (3.10) are conveniently represented in terms of the variables $\tau = \xi/c$ and $z = (u_c'/2B)^{1/2}(Y - Y_c(\tau))$, where $Y_c(\tau) = (\Omega - \Theta_\tau)/(ku_c')$ is the displacement of a critical level of the disturbance under consideration with respect to

† At least for the case $P_m \sim 1$. For $P_m \ll 1$, the situation is somewhat more complicated (see § 6.3).

‡ Recall that the quasi-steadiness can occur not only in a developed nonlinear CL, but also in the viscous CL regime the quasi-stationarity conditions are always satisfied (Churilov & Shukhman 1996).

a critical level of the neutral mode. We obtain

$$\mathscr{M}_{(\lambda/P_m)}\phi_m = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}\phi_m, \tag{5.2}$$

$$\mathscr{M}_\lambda\zeta_m = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}\zeta_m + \frac{u_c'}{(2B)^2}\left(\phi_m'\frac{\partial\phi_m''}{\partial\theta} - \phi_m'''\frac{\partial\phi_m}{\partial\theta}\right), \tag{5.3}$$

$$\mathscr{M}_\lambda\zeta_h = -\frac{1}{k(2Bu_c')^{1/2}}\mathscr{T}\zeta_h - \frac{2u_c'''}{k(2Bu_c'^3)^{1/2}}\left[(\Omega-\Theta_\tau)B\sin\theta + B_\tau\cos\theta\right] , \tag{5.4}$$

where

$$\mathscr{M}_\mu = z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z} - \mu\frac{\partial^2}{\partial z^2}, \quad \mathscr{T} = \frac{\partial}{\partial\tau} + \frac{\Theta_{\tau\tau}}{k(2Bu_c')^{1/2}}\frac{\partial}{\partial z} - \frac{B_\tau}{2B}z\frac{\partial}{\partial z}, \quad \lambda = \frac{\eta(u_c')^{1/2}}{k(2B)^{3/2}} ,$$

the prime designates the derivative with respect to z, $\phi_m \equiv \phi + C_A\left(2B/u_c'\right)^{1/2} z$ and λ is so-called Haberman parameter which characterizes the relative importance of viscosity and nonlinearity. In the first approximation unsteady terms (terms with $\mathscr{T}$ on the right-hand sides of (5.2)–(5.4)) can be neglected. It is in this sense that we are using the term *quasi-steadiness* throughout: ζ and ϕ depend in this case on the evolution variable only parametrically – through a local instantaneous value of the amplitude.

In Churilov & Shukhman (1996), devoted to quasi-steady evolution regimes, we introduced four functions of the Haberman parameter λ which can be used to represent the NEE in such regimes. Proceeding here along the same lines, we introduce a further new function $\Phi_5(\lambda, P_m)$:

$$\Phi_5(\lambda, P_m) = \fint_{-\infty}^{\infty}\langle g_5\cos\theta\rangle \mathrm{d}z, \tag{5.5}$$

where $g_5(\lambda, P_m; z, \theta)$ satisfies the equation

$$\mathscr{M}_\lambda g_5 = \frac{1}{4}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial\theta}\left(f_1'\right)^2 - \left(f_1'\frac{\partial f_1}{\partial\theta}\right)'\right]. \tag{5.6}$$

Here $f_1 = g_1(\lambda/P_m; z,\theta) + 2z$, $g_1(\lambda; z, \theta)$ is the solution of the equation

$$\mathscr{M}_\lambda g_1(\lambda; z, \theta) = -2\sin\theta \tag{5.7}$$

with boundary condition $\partial g_1/\partial z \to 0$ as $z \to \pm\infty$. Note that it immediately follows from the symmetry properties of (5.6), (5.7) that $\int\langle g_5\sin\theta\rangle \mathrm{d}z = 0$.

Using the functions g_5 and g_1, and also the function $g_2(\lambda; z, \theta)$ introduced in Churilov & Shukhman (1996) and as the solution of the equation $\mathscr{M}_\lambda g_2 = -2\cos\theta$, the solution of the system (5.2)–(5.4) is written as

$$\phi_m = C_A\left(\frac{B}{2u_c'}\right)^{1/2}\left[g_1\left(\lambda/P_m; z,\theta\right) + 2z\right] + \cdots \equiv C_A\left(\frac{B}{2u_c'}\right)^{1/2} f_1 + \cdots , \tag{5.8}$$

$$\zeta_m = \frac{C_A^2}{2B}\, g_5(\lambda, P_m; z, \theta) + \cdots , \tag{5.9}$$

$$\zeta_h = \frac{u_c'''}{k(2Bu_c'^3)^{1/2}}\left[(\Omega - \Theta_\tau)Bg_1(\lambda; z,\theta) + B_\tau g_2(\lambda; z, \theta)\right] + \cdots . \tag{5.10}$$

Dots designate contributions caused by unsteady terms. The function $g_5(\lambda, P_m; z, \theta)$ and the associated function $\Phi_5(\lambda, P_m)$ (for $P_m = O(1)$) are calculated in Appendix A.

At this point we shall give only the asymptotic expressions for Φ_5 at small (and also large) values of λ needed now:

$$\Phi_5(\lambda, P_m) \overset{P_m=O(1)}{=} \begin{cases} C^{(5)} P_m^{-1/2} \lambda^{-1/2}, & \lambda \ll 1, \\ a_5(P_m) \lambda^{-7/3}, & \lambda \gg 1. \end{cases} \tag{5.11}$$

Here

$$C^{(5)} = \frac{2}{3} \int_0^\infty \frac{\mathrm{d}z}{z^{3/2}} \left(\frac{1}{\mathrm{e}^z - 1} - \frac{1}{z} + \frac{1}{2} \right) = 0.49129\ldots,$$

and the function $a_5(P_m)$ is plotted in figure 3.

Contributions to the right-hand sides of MSC (3.8) are expressed in terms of the functions $\Phi_1(\lambda)$, $\Phi_2(\lambda)$ and a newly introduced function $\Phi_5(\lambda, P_m)$ as

$$\fint_{-\infty}^{\infty} \langle \zeta_h \sin\theta \rangle \mathrm{d}Y = \frac{u_c'''}{k u_c'^2} \Phi_1(\lambda) \left(\Omega - c \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} \right) B, \quad \fint_{-\infty}^{\infty} \langle \zeta_h \cos\theta \rangle \mathrm{d}Y = \frac{u_c'''}{k u_c'^2} \Phi_2(\lambda)\, c \frac{\mathrm{d}B}{\mathrm{d}\xi},$$

$$\fint_{-\infty}^{\infty} \langle \zeta_m \cos\theta \rangle \,\mathrm{d}Y = \frac{c_A^2}{\left(2Bu_c'\right)^{1/2}} \Phi_5(\lambda, P_m)\,, \qquad \fint_{-\infty}^{\infty} \langle \zeta_m \sin\theta \rangle \,\mathrm{d}Y = 0\,. \tag{5.12}$$

The 'magnetic' function $g_5(\lambda, P_m; z, \theta)$, like the 'stratified' function $g_4(\lambda, Pr; z, \theta)$ of I, is even, i.e. it is invariant under the transformation $z \to -z$, $\theta \to 2\pi - \theta$. Therefore, it, like g_4, makes only a '$\cos\theta$' contribution to the MSC. Furthermore, as in I (and in contrast to the case of weakly three-dimensional disturbances (Churilov & Shukhman 1995)), the interaction of the regular and singular components occurs throughout the CL thickness, and not only in outer diffusion layers, and hence this contribution has a local (i.e. dependent on the value of the amplitude in a given place only), rather than integral character. These factors are both of important significance: because of the identity of the symmetry properties and the character of the interaction between the regular and singular components, the corresponding NEE and hence the evolution character turn out also to be alike in the two problems.

Substituting (5.12) into the right-hand sides of (3.8) and solving the resulting system for $\mathrm{d}B/\mathrm{d}\xi$, gives

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \frac{2k^2 u_c'''}{u_c'^2} \left[I_2 |\Omega| B - \frac{c}{2k} \frac{C_A^2}{\left(2Bu_c'\right)^{1/2}} \Phi_5(\lambda, P_m) \right] \frac{\Phi_1(\lambda)}{\Delta(\lambda)}, \tag{5.13}$$

where $\Delta(\lambda) \equiv I_0^2 + c^2 (u_c'''/u_c'^2)^2 \Phi_1(\lambda) \Phi_2(\lambda)$.

A NEE in the form (5.13) is suitable for describing both quasi-steady regimes, i.e. not only the nonlinear ($\lambda \ll 1$) but also the viscous ones ($\lambda \gg 1$). When $\lambda \gg 1$ it coincides with equation (4.5) written in the real form, and when $\lambda \ll 1$ it takes the form

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \eta \frac{C^{(1)}}{\Delta(0)} \frac{u_c'''}{u_c'} \left[\frac{k I_2 |\Omega|}{(2Bu_c')^{1/2}} - \frac{c k^{1/2} c_A^2}{(\eta P_m)^{1/2}} \frac{C^{(5)}}{(2Bu_c')^{5/4}} \right]. \tag{5.14}$$

Equation (5.14) describes the development of disturbances in the regime of a developed nonlinear CL, provided that this regime can be reached evolutionarily, from the initially small disturbance.

In the subsequent analysis, we pass to 'physical' variables and represent (5.14) in brief form:

$$\frac{\mathrm{d}|A|}{\mathrm{d}x} = \left(d_2 \frac{\nu \gamma_L}{|A|^{3/2}} - \frac{d_3}{P_m^{1/2}} \frac{c_A^2 \nu^{1/2}}{|A|^{9/4}} \right) |A|, \quad d_2 > 0, \quad d_3 > 0. \tag{5.15}$$

On evaluating from (5.15) the nonlinear growth rate $\gamma = |A|^{-1}\mathrm{d}|A|/\mathrm{d}x$ as

$$\gamma \sim \max\left\{\frac{\nu\gamma_L}{|A|^{3/2}}, \frac{c_A^2\nu^{1/2}}{|A|^{9/4}}\right\},$$

we find that the quasi-steadiness conditions (5.1), i.e. the validity range of (5.14), (5.15), are satisfied if

$$\max\left\{\frac{\nu\gamma_L}{|A|^{3/2}}, \frac{c_A^2\nu^{1/2}}{|A|^{9/4}}\right\} \ll \frac{\nu}{|A|} \ll |A|^{1/2}, \quad \text{or} \quad |A| \gg \max\left\{c_A\left(\frac{c_A}{\nu^{2/3}}\right)^{3/5}, \gamma_L^2, \nu^{2/3}\right\}. \tag{5.16}$$

In the subsequent analysis, once again, it is necessary to distinguish between the two cases: $c_A > \nu^{2/3}$ and $c_A < \nu^{2/3}$. In figures 6 and 7 (referring to the cases $c_A > \nu^{2/3}$ and $c_A < \nu^{2/3}$) the region where the quasi-steadiness conditions in the nonlinear CL regime (5.16) are satisfied is shown by cross-hatching.

6. The resulting evolution scenario

We now endeavour to sketch a full evolution picture based on the results obtained in §§4 and 5. At this point our concern is with the fate of only those disturbances which do not reach saturation in the viscous CL regime.

6.1. *The case* $c_A \gg \nu^{2/3}$

A distinguishing characteristic of this case is the presence of a 'gap' (filled in figure 6)

$$c_A < |A| < c_A\left(c_A/\nu^{2/3}\right)^{3/5}, \quad \gamma_L^2 < |A|, \tag{6.1}$$

between the amplitude level where the transition to the nonlinear CL regime occurs, and the validity range of equation (5.15) (i.e. the region where the approximation of a quasi-steady nonlinear CL holds true). This gap generates a break in the continuous description of the evolution. To understand what happens to the disturbance after it enters this gap from the explosive growth stage, it is necessary to have the numerical solution of the 'exact' system of equations (3.6)–(3.8)†. We came across precisely the same situation in I (the case $Pr < 1$) where we embarked on such a numerical study, it merely lent support to a qualitative picture which could be predicted, based on analysing the equation obtained for the quasi-steady nonlinear CL regime. Thus we shall not give here unwieldy numerical calculations but make use of the analogy.

The magnetic contribution in (5.15) is negative, which corresponds to a decrease of the amplitude in the amplitude region where this contribution dominates, i.e. when

$$|A| < A_4 \sim \left(c_A^2/\nu^{1/2}\gamma_L\right)^{4/3}. \tag{6.2}$$

Hence when $\gamma_L < c_A^{1/2}\left(c_A/\nu^{2/3}\right)^{3/10}$ (see figure 6) the region of a quasi-steady nonlinear CL is evolutionarily unattainable: even though the disturbance approaches it from below, it will be expelled backward (which is symbolized by the $\bigcap$-shaped arrow in figure 6). This means that the amplitude of disturbances entering this gap will

† In fact, immediately after the stage of the explosive growth (4.11) the unsteady and nonlinear terms in (3.6) and (3.7) become dominant and of the same order: $\ell_t \sim \ell_N \gg \ell_\nu$. This does not permit us to investigate the evolution by analytical methods. For the numerical solution in the gap, the initial condition resulting from matching with the previous stages is needed. It is very difficult to make such a matching in reality, hence even if we tackled this problem numerically it would be best to start from the very beginning, i.e. from a very small amplitude, as done in I.

oscillate in a limited range, without exceeding the level corresponding to the gap's upper boundary:

$$|A|_{max} \sim c_A \left(c_A/\nu^{2/3}\right)^{3/5}. \tag{6.3}$$

It is this behaviour that we observed in numerical calculations done in I pertaining to a similar situation ($Pr < 1$, see figure 13 in I). From this standpoint one can conclude that nonlinearity ultimately inhibits the amplitude not only in the region of supercriticalities corresponding to a viscous CL ($\gamma_L < \nu^{1/3}$) but also in the adjacent region of explosive growth in the unsteady CL regime ($\nu^{1/3} < \gamma_L < c_A^{1/2}\left(c_A/\nu^{2/3}\right)^{3/10}$).

In the region of supercriticalities $\gamma_L > c_A^{1/2}\left(c_A/\nu^{2/3}\right)^{3/10}$ the magnetic field scarcely affects the evolution, and the latter proceeds almost in the same manner as in non-conducting fluid: following a relaxation stage in the region of amplitudes $|A| \sim A_3 = \gamma_L^2$ the exponential growth is substituted by a power-law growth

$$|A| \propto (\nu\gamma_L)^{2/3} x^{2/3} \tag{6.4}$$

in the regime of a developed quasi-steady nonlinear CL (Goldstein & Hultgren 1988).

6.2. *The case $c_A \ll \nu^{2/3}$*

The evolution picture for this case is presented schematically in figure 7. There is no unsteadiness gap in the nonlinear CL regime in this case, and equation (5.13) would suffice to describe the entire course of evolution in the region $\gamma_L < \nu^{1/3}$.

As has already been pointed out, in the region of the viscous CL ($|A| < \nu^{2/3}, \gamma_L < \nu^{1/3}$) equation (4.5) applies and describes the stabilization on the level $A_{sat} \sim \left(\nu^{7/3}\gamma_L/c_A^2\right)^{1/2}$. With increasing γ_L (to the value of c_A^2/ν), A_{sat} grows and approaches the nonlinear CL boundary where a description of the evolution using equation (4.5) is no longer applicable. In this region the saturation level can be determined from an exact equilibrium condition obtainable by setting the right-hand side of (5.13) equal to zero:

$$\Phi(\lambda, P_m) \equiv \lambda\Phi_5(\lambda, P_m) = I_2|\Omega|\eta u_c'/(cC_A^2), \quad \Phi(\lambda, P_m) = \begin{cases} \left(\lambda/P_m\right)^{1/2} C^{(5)}, & \lambda \ll 1, \\ a_5(P_m)\lambda^{-4/3}, & \lambda \gg 1. \end{cases} \tag{6.5}$$

The function Φ has a maximum of order unity at λ of order unity. Hence it follows from (6.5) that when $|\Omega| > |\Omega|_* \sim O\left(C_A^2/\eta\right)$, or, in physical variables, when $\gamma_L > (\gamma_L)_* \sim O\left(c_A^2/\nu\right)$ there exist no equilibrium solutions. This means that the magnetic field cannot stop the growth of such disturbances, and when $|A| \sim O\left(\nu^{2/3}\right)$ (for $\gamma_L < \nu^{1/3}$) or when $|A| \sim O\left(\gamma_L^2\right)$ (for $\gamma_L > \nu^{1/3}$) they reach the quasi-steady nonlinear CL regime with power-law growth (6.4).

In closing it may be noted that the regime of a developed quasi-steady nonlinear CL ($\lambda \ll 1$) described by equation (5.14), in which the magnetic term would play *an active role*, is not realized in any of the cases considered – either when $c_A > \nu^{2/3}$, or when $c_A < \nu^{2/3}$: the disturbance either reaches a stable state at smaller amplitudes ($\lambda \gtrsim 1$) or passes to the *unsteady* nonlinear CL regime.

6.3. *Evolution at small magnetic Prandtl numbers, $P_m \ll 1$*

In the foregoing discussion we confined ourselves to studying a flow with $P_m = O(1)$. With dissipation coefficients differing greatly from each other, there emerges an ever more crowded (with different variants) evolution picture, because we now have two scales rather than one ℓ_ν: a viscous scale ℓ_ν and a magneto-viscous scale ℓ_{ν_m}. The number of possible relations between the value of the magnetic field and dissipation coefficients also increases: instead of two ($c_A < \nu^{2/3}$ and $c_A > \nu^{2/3}$). we now have three. A description of research results on all possible variants arising if we abandon the condition $P_m = O(1)$ would make our paper too unwieldy. Hence we restrict ourselves only to a brief presentation of results relating to the case $P_m \ll 1$ (i.e. $\nu \ll \nu_m$) and consider only two of the three possible relations between c_A, ν and ν_m, namely: $c_A \ll \nu^{2/3}$ and $c_A \gg \nu_m^{2/3}$.

6.3.1. *The viscous and unsteady CL regimes*

As long as the amplitude is small enough, the evolution is described by equation (4.1) (or by (4.3)). To appreciate the way in which disturbances develop in these regimes when $\nu_m \gg \nu$, we consider three limiting cases:

$$\begin{array}{lll} \text{(i)} & \gamma \ll \nu^{1/3} & (\text{i.e. } \ell_t \ll \ell_\nu \ll \ell_{\nu_m}), \\ \text{(ii)} & \nu^{1/3} \ll \gamma \ll \nu_m^{1/3} & (\text{i.e. } \ell_\nu \ll \ell_t \ll \ell_{\nu_m}), \\ \text{(iii)} & \nu_m^{1/3} \ll \gamma \ll 1 & (\text{i.e. } \ell_{\nu_m} \ll \ell_t). \end{array}$$

For each of these cases, from (4.3) we obtain the corresponding NEE:

$$\text{(i)}\ \frac{\mathrm{d}A}{\mathrm{d}x} - \gamma_L A = \frac{4\mathrm{e}^{-\mathrm{i}\varphi}}{|\tilde{I}_0|} k^{10/3} u_c'^{-5/3} c_A^2 \frac{|a_2|}{\nu^{4/3}\nu_m} A|A|^2, \tag{6.6}$$

$$\text{(ii)}\ \frac{\mathrm{d}A}{\mathrm{d}x} - \gamma_L A = \frac{2\pi\mathrm{e}^{-\mathrm{i}\varphi}}{|\tilde{I}_0|} \frac{k^6 u_c' c_A^2}{\nu_m c^4} \int_0^\infty \mathrm{d}s\, s^3 \int_0^1 \mathrm{d}\sigma\, \sigma^2 A(x-s) A(x-\sigma s) \overline{A}(x-(1+\sigma)s), \tag{6.7}$$

$$\text{(iii)}\ \frac{\mathrm{d}A}{\mathrm{d}x} - \gamma_L A = \frac{4\pi\mathrm{e}^{-\mathrm{i}\varphi}}{3|\tilde{I}_0|} \frac{k^8 u_c'^3 c_A^2}{c^7} \int_0^\infty \mathrm{d}s\, s^6 \int_0^1 \mathrm{d}\sigma\, \sigma^3 (1+\sigma^2) A(\cdots) A(\cdots) \overline{A}(\cdots). \tag{6.8}$$

Equation (6.6) trivially follows from (4.5) (if we make use of the asymptotic expression (4.6) for $a_5(P_m)$ at $P_m \ll 1$). It, like (4.5), gives stabilization at $|A| = A_{sat}$:

$$A_{sat} = \tilde{d}_1 \left(\frac{\nu^{4/3} \nu_m \gamma_L}{c_A^2} \right)^{1/2}, \quad \tilde{d}_1 = \left[\frac{u_c'^{5/3} |\tilde{I}_0|}{8k^{10/3} a_5^{(s)} |\cos\varphi|} \right]^{1/2}. \tag{6.9}$$

Equation (6.8) describing the evolution in the unsteady CL regime, coincides with (4.8) and gives an explosive growth $|A| \propto c_A^{-1}(x_0 - x)^{-4}$. A substantially new element is equation (6.7) referring to the intermediate region when the unsteady scale ℓ_t is larger than the viscous scale ℓ_ν but smaller than the magneto-viscous scale ℓ_{ν_m}. It turns out that in this case, as in the case of the usual unsteady CL, the equation retains its non-local character and after the amplitude reaches the nonlinearity threshold $|A| \sim A_5$,

$$A_5 \sim \nu_m^{1/2} \gamma_L^{5/2} / c_A\ ,$$

also gives an explosive growth:

$$|A| \propto \frac{\nu_m^{1/2}}{c_A} (x_1 - x)^{-5/2}. \tag{6.10}$$

It is interesting to note a structural change of the nonlinear term with increasing $\ell_t \sim \gamma$, i.e. at the transition (i) $\rightarrow$ (iii):

$$c_A^2 \frac{A^3}{\ell_{\nu_m}^3 l_\nu^4} \rightarrow c_A^2 \frac{A^3}{\ell_{\nu_m}^3 \ell_t^4} \rightarrow c_A^2 \frac{A^3}{\ell_t^7}.$$

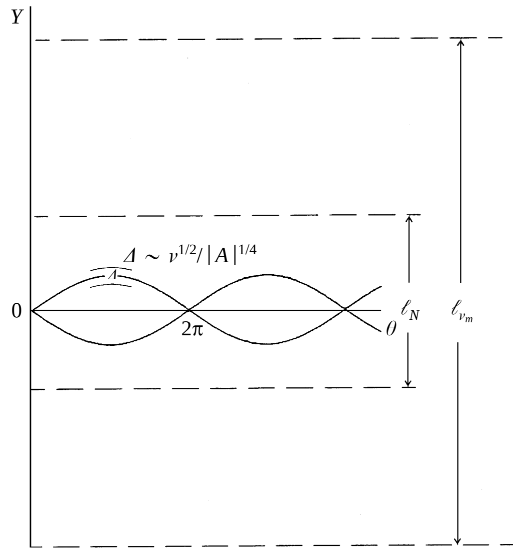

FIGURE 8. Schematic representation of the CL structure when $P_m \ll 1$ and $\ell_\nu \ll \ell_N \ll \ell_{\nu_m}$. Δ is the 'width' of the separatrix (i.e of the 'cat's eye' boundary).

Such a structure is easy to see by analysing the structure of equation (4.5) and the dependence of the Landau constant on P_m. Therefore, both the structure of NEE (6.6)–(6.8) and hence the law of evolution (6.10) can be readily 'guessed right' based on such analysis only.

6.3.2. *Quasi-steady CL regimes*

As in the case $P_m = O(1)$, explosive stages are intermediate ones. To construct a full picture, it is necessary to supplement the set of NEE (6.6)–(6.8) with an equation suitable for description of quasi-steady CL regimes. Now instead of two limiting cases – $|A| \ll \nu^{2/3}$ ($\lambda \gg 1$) and $|A| \gg \nu^{2/3}$ ($\lambda \ll 1$) – one needs to consider three:

$$\begin{array}{lll} (a) & |A| \ll \nu^{2/3} & (\lambda \gg 1, \quad \ell_N \ll \ell_\nu), \\ (b) & \nu^{2/3} \ll |A| \ll \nu_m^{2/3} & (P_m \ll \lambda \ll 1, \quad \ell_\nu \ll \ell_N \ll \ell_{\nu_m}), \\ (c) & \nu_m^{2/3} \ll |A| & (\lambda \ll P_m, \quad \ell_{\nu_m} \ll \ell_N). \end{array}$$

The CL can now be called truly viscous (or, more properly, dissipative) only in case (a) and truly nonlinear in case (c). Case (b) is an intermediate one: the nonlinear CL scale $\ell_N = |A|^{1/2}$ is now larger than the viscous scale ℓ_ν, but is still smaller than the magneto-viscous scale $\ell_{\nu_m} = \nu_m^{1/3}$. By this it is meant that the magnetic field structure obeying equation (3.7) is described by the magneto-viscous CL approximation and can be calculated by a well-known method. The structure of the vorticity field ζ_m determined from (3.7) is more complicated, however. It constitutes a chain of 'cat's eyes' with the transverse size $\sim \ell_N$ embedded in a wide region of the magneto-viscous CL (figure 8).

As a result of calculations, we again arrive at the NEE with the same form as (5.13); however, at $P_m \ll 1$ a different form of the asymptotic expansion of the function

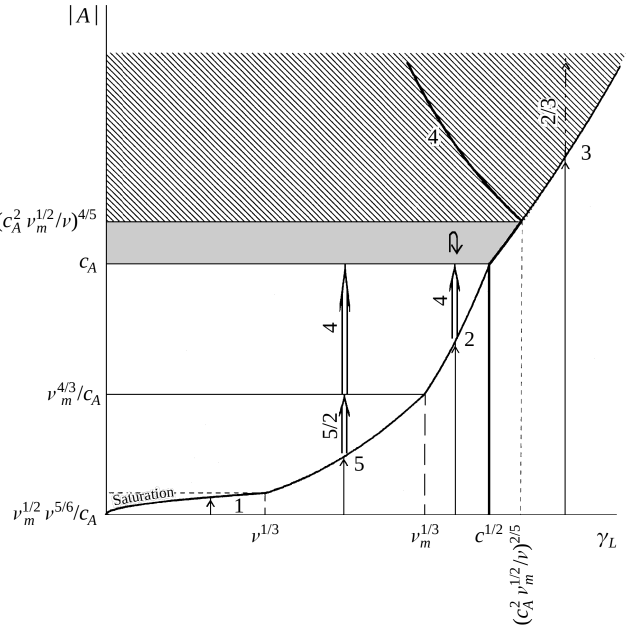

FIGURE 9. Diagram of the development of disturbances at $P_m \ll 1$. The case $c_A \gg \nu_m^{2/3}$. Moderately heavy lines show nonlinearity thresholds: curve 1, $A_1 = (\gamma_L \nu_m \nu^{4/3}/c_A^2)^{1/2}$; 2, $A_2 = \gamma_L^4/c_A$; 3, $A_3 = \gamma_L^2$; 5, $A_5 = \nu_m^{1/2}\nu^{2/3}\gamma_L^{1/2}/c_A$. Curve 4, $A_4 = (\nu_m^{1/2}c_A^2/\nu\gamma_L)^{4/3}$. The meaning of the other designations is the same as in figure 6.

$\Phi_5(\lambda, P_m)$, obtained in Appendix B, must be used here:

$$\Phi_5(\lambda, P_m) \overset{P_m \ll 1}{=} \begin{cases} C^{(5)} P_m^{-1/2}\lambda^{-1/2}, & \lambda \ll P_m, \\ \frac{1}{2}|C^{(2)}|P_m \lambda^{-2}, & P_m \ll \lambda \ll 1, \\ \frac{1}{2}|a_2|P_m \lambda^{-7/3}, & \lambda \gg 1, \end{cases} \tag{6.11}$$

where $C^{(2)} = -2.5008\ldots$, $a_2 = -1.6057\ldots$.

From this equation at $\lambda \gg 1$ we again obtain equation (6.6) and at $\lambda \ll P_m$ equation (5.14) which describes the evolution in the quasi-steady nonlinear CL regime (which is evolutionarily unattainable here, as in the case with $P_m = O(1)$). In the region $P_m \ll \lambda \ll 1$, from (5.13) and (6.11) we obtain the NEE which we write in brief form in physical variables:

$$\frac{\mathrm{d}|A|}{\mathrm{d}x} = \left(d_2 \frac{\nu\gamma_L}{|A|^{3/2}} - d_4 \frac{c_A^2}{\nu_m}\right)|A|, \quad d_4 > 0. \tag{6.12}$$

This equation, like (6.6), has a stable steady solution

$$|A|_{sat} = A_6 \sim \left(\nu\nu_m\gamma_L/c_A^2\right)^{2/3} . \tag{6.13}$$

The set of NEEs (6.6)–(6.8), (5.14) and (6.12) permits us to describe the development of disturbances at $P_m \ll 1$. This can best be done employing the amplitude–supercriticality diagram. As mentioned above, we shall confine our treatment to two variants only: $c_A \gg \nu_m^{2/3}$ and $c_A \ll \nu^{2/3}$.

The evolution when $c_A \gg \nu_m^{2/3}$ is shown in figure 9. The picture in this case resembles that obtained in our investigation of the case $c_A \gg \nu^{2/3}$ at $P_m = O(1)$

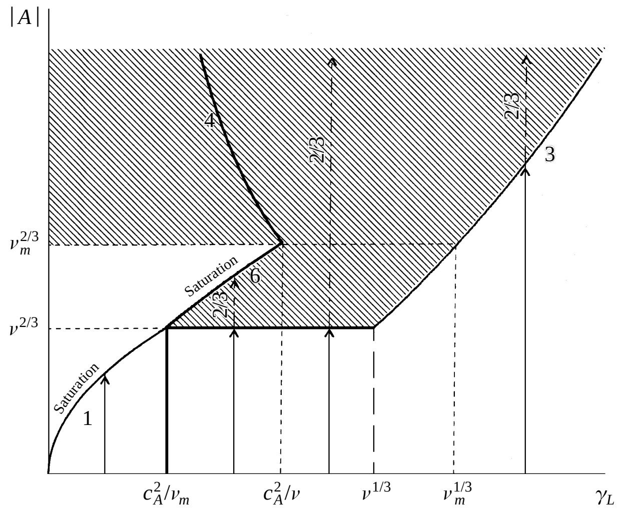

FIGURE 10. Same as figure 9 but for the case $c_A \ll \nu^{2/3}$. Here curve 6, $A_6 = (\gamma_L \nu_m \nu / c_A^2)^{2/3}$.

(figure 6). A new element is the presence of an intermediate asymptotic representation when $\nu^{1/3} < \gamma_L < \nu_m^{1/3}$ in the region of amplitudes $\nu_m^{1/2}\nu^{5/6}/c_A < |A| < \nu_m^{4/3}/c_A$, namely: $|A| \sim (x_1 - x)^{-5/2}$. As in the case with $P_m = O(1)$, there exists a region of supercriticalities, $\nu^{1/3} < \gamma_L < (c_A^2 \nu_m^{1/2}/\nu)^{2/5}$, where the growth of disturbances is limited:

$$|A| < A_{max} \sim \left(c_A^2 \nu_m^{1/2}/\nu\right)^{4/5},$$

but the stationary solution is not established here, unlike the region of supercriticalities $\gamma_L < \nu^{1/3}$, where $|A|$ tends to an equilibrium value (6.13). When $\gamma_L > (c_A^2 \nu_m^{1/2}/\nu)^{2/3}$ the magnetic field does not affect the evolution, and when $|A| > \gamma_L^2$ the transition to the regime of power-law growth $|A| \sim x^{2/3}$ occurs here.

The evolution in the case of a very weak magnetic field $c_A \ll \nu^{2/3}$ is shown in figure 10. A substantially new element here (compared with the case $P_m = O(1)$) is the presence in the viscous CL region of a subregion

$$c_A^2/\nu_m < \gamma_L < c_A^2/\nu, \quad \nu^{2/3} < |A| < \nu_m^{2/3},$$

in which the evolution when $|A| > \nu^{2/3}$ proceeds in accordance with equation (6.12). Note that the non-magnetic term gives the main contribution to the right-hand side of (6.12) when $|A| < |A|_{sat}$. Therefore, the disturbance, upon reaching the level $|A| \sim \nu^{2/3}$, passes to the usual regime of the quasi-steady nonlinear CL with the law $|A| \sim x^{2/3}$, and only after that, at larger amplitudes, the magnetic nonlinearity comes into play, which leads to the stabilization.

In the region $\gamma_L > c_A^2/\nu$ there are no differences from the case $P_m = O(1)$ (cf. figure 7).

To conclude this Section, we wish to point out that in the case of large P_m, $P_m \gg 1$, which was also investigated by this author, there exists no stable stationary solution in the intermediate region $\nu_m^{2/3} < \gamma_L < \nu^{2/3}$, similar to (6.13). For completeness we also give here the results derived by calculating the function $\Phi_5(\lambda, P_m)$ at large P_m:

$$\Phi_5(\lambda, P_m) \overset{P_m \gg 1}{=} \begin{cases} C^{(5)} P_m^{-1/2} \lambda^{-1/2}, & \lambda \ll P_m^{-1}, \\ c_5, & P_m^{-1} \ll \lambda \ll 1, \\ b_5 \lambda^{-2/3}, & 1 \ll \lambda \ll P_m, \\ a_5^{(l)} P_m^{5/3} \lambda^{-7/3}, & P_m \ll \lambda, \end{cases}$$

where $c_5 = 0.5553\ldots$, $b_5 = 0.3523\ldots$, $a_5^{(l)} = 0.58154\ldots$.

7. Discussion

This treatment has generally reinforced the anticipated evolution picture typical of disturbances with the weakly singular neutral mode, enabling the determination for the flow at hand of the parameter which governs the transition from the 'fast' to 'slow' scenario. Without going into the details associated with a possible large difference of dissipative coefficients and complicating the overall picture, we summarize the results obtained assuming that $P_m = O(1)$.

It turns out that in a flow with a weak parallel magnetic field the key role is played by the parameter $c_A/\nu^{2/3}$. Like the parameter $Ri/\nu^{2/3}$ in the weakly stratified flow problem (I) and the parameter $k_z^4/\nu^{2/3}$ in the problem of weakly three-dimensional disturbances (Churilov & Shukhman 1995), it is responsible for the realization of two different types of evolution behaviour: when $c_A \gg \nu^{2/3}$ there exists a range of supercriticalities, $\nu^{1/3} < \gamma_L < c_A^{1/2}$, where explosive behaviour occurs, and when $c_A \ll \nu^{1/3}$ the region of explosive behaviour disappears.

The analysis also revealed the stabilizing role of the magnetic nonlinearity. The manifestation of this role is different for $c_A < \nu^{2/3}$ and $c_A > \nu^{2/3}$. When $c_A < \nu^{2/3}$ this role simply implies that the magnetic nonlinearity in the part of the viscous CL region where it is still important ($\gamma_L < c_A^2/\nu$)), stabilizes the instability in the viscous CL regime ($|A| < \nu^{2/3}$) and prevents the transition to the nonlinear quasi-steady CL regime (figure 7).

When $c_A > \nu^{2/3}$ the stabilizing role of the magnetic nonlinearity manifests itself in a greater variety of fashions. Thus, all disturbances starting from the viscous CL region ($\gamma_L < \nu^{1/3}$) reach saturation in the viscous CL regime ($|A| < \nu^{2/3}$), with the nonlinear evolution ceasing. The behaviour of disturbances starting from the unsteady CL region ($\gamma_L > \nu^{1/3}$) is more complicated. The growth of disturbances with $\nu^{1/3} < \gamma_L < c_A^{1/2}$ upon reaching the nonlinearity threshold evolves into an explosive phase, $|A| \sim (x_0 - x)^{-4}$, and right up to the transition to the nonlinear CL regime at $|A| \sim c_A$ the stabilizing role of the magnetic nonlinearity hardly manifests itself at all (except for the appearance of local parts where the amplitude decreases (see figure 5) which are absent in the case of a destabilizing sign of the nonlinearity (e.g. figure 3 of I). However, upon reaching the nonlinear CL regime and entering the 'gap' region (figure 6), the disturbance ceases to grow and begins to decrease, with quasi-periodic oscillations setting in subsequently. The behaviour of disturbances with $c_A < \gamma_L < c_A^{1/2}(c_A/\nu^{2/3})^{3/10}$ is similar to that of disturbances with $\nu^{1/3} < \gamma_L < c_A^{1/2}$ with the only difference that they enter the nonlinear CL region not from the explosive growth stage but immediately from the exponential growth stage.

The stabilizing effect of the magnetic nonlinearity ceases to have influence for the perturbations with larger supercriticality only: $c_A^{1/2}(c_A/\nu^{2/3})^{3/10} < \gamma_L \ll 1$, which evolve as in the case of a flow of non-conducting liquid.

A principal goal of this paper was to elucidate the question: to what extent is the ‘gap’ in the unsteady nonlinear CL (figure 6) after the explosive stage in the unsteady (linear) CL regime a typical element of the evolution scenario? This question may also be formulated thus: what are the characteristic features of the interaction between the regular and singular disturbance components which lead to the fact that the amplitude at which the nonlinear CL becomes nonlinear ($\ell_N > \ell_t, \ell_\nu$) is smaller than the amplitude at which the nonlinear CL becomes also quasi-steady ($\ell_t < \ell_\nu^3/\ell_N^2 < \ell_N$), i.e. to the fact that there exists a range of amplitudes (gap) where the CL is already a nonlinear one, but it cannot yet be described in terms of a quasi-steady approximation (terms with $\mathscr{T}$ in equations (5.2)–(5.4) cannot be discarded)?

We have tried in I (where we detected for the first time the presence of the ‘gap’) to understand in ‘physical language’ the reason behind its behaviour. The presence of the ‘gap’ is due to the fact that, because of the back influence of the regular component (vorticity) upon the singular component (density in I, or the magnetic vector-potential ϕ here), a refinement of the scales accompanying the transition to the nonlinear CL regime and related to the non-isochronic motion of neighbouring liquid particles, ceases on a scale L which is larger than the scale $L_D \sim \nu/|A|$ on which the diffusion comes into play (leading to a fast smoothing and the formation of a quasi-steady nonlinear CL).

By comparing the findings reported by Churilov & Shukhman (1995), in I, and in this paper, we can now suggest that the unsteady nonlinear CL stage following the explosive growth stage in the unsteady CL regime is more likely to be a typical element of the evolution scenario for disturbances with the weakly singular neutral mode rather than the exception attributed to some degeneration. In the case of the problem of a weakly stratified flow this degeneracy implied that, because of the specific symmetry of the equations, the nonlinear term in the NEE caused by the stratification turned out to be reduced, as compared with its prior estimate, to the factor $(\ell_\nu/\ell)^3$, where ℓ is the CL scale, and one would suspect that this is precisely the reason for the presence of the gap. In this paper the formal reason for the appearance of the gap that when $\lambda \ll 1$ the function $\Phi_5 = O(\lambda^{-1/2}) \gg 1$ rather than $O(1)$†, and such a behaviour of Φ_5 is dictated by the character of interaction between the singular and regular disturbance components and is unassociated with degeneracy. By the character of interaction of the components we mean the following: either it occurs throughout the CL, as is the case here and in I, or it is expelled to the periphery, to the region of outer diffusion layers, as was the case in the problem of weakly three-dimensional disturbances (Churilov & Shukhman 1995), as well as the symmetry properties of the components. It will be recalled that in the present case, as in I, the regular component is even with respect to the transformation $z \to -z,\ \theta \to 2\pi - \theta$ and hence it makes only a ‘cosine’ contribution to the MSC (see (5.12)), while in the problem of weakly three-dimensional disturbances it is odd and, accordingly, makes only a ‘sine’ contribution to the MSC (Churilov & Shukhman 1995). The gap is most likely to occur if the interaction of the components has the same character as in the present case and in I.

In closing we wish to note a further aspect of the problem at hand. As we have repeatedly emphasized, in the case of a sufficiently large supercriticality the magnetic field plays a passive role only and the amplitude increases in the same manner as in

† It is easy to see that if at $\lambda \ll 1$ we had $\Phi_5 = O(1)$, it would follow from (5.13) that the boundary of the transition to the quasi-steadiness regime would coincide with the boundary of the transition to the nonlinear CL $|A| \sim c_A$.

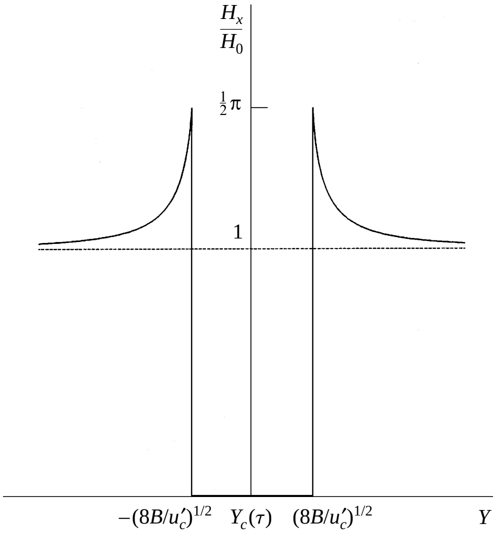

FIGURE 11. Profile of the x-component of the magnetic field in the midsection ($\theta = \pi$).

the case without a field (i.e. $|A| \propto (\nu\gamma_L)^{2/3}x^{2/3}$) right up to the applicability limits of weakly nonlinear theory ($|A| \sim 1$) (unless factors neglected here come into play still earlier, such as the viscous expansion of an undisturbed flow (Goldstein & Hultgren 1988)). However, the magnetic field itself is affected by the velocity field. If the initial magnetic field $H_0 = O(\varepsilon)$ is considered to be a certain 'priming' field, we may regard our problem as a kinematic dynamo problem. In this case it is appropriate to pose the question of the effectiveness of such a dynamo, i.e. to what extent can the initial magnetic field be enhanced in the course of the nonlinear development of a shear flow instability. It is easy to see that such a dynamo mechanism does not give a significant enhancement of the field. Indeed, in the regime of a developed nonlinear CL ($\lambda \to 0$) the field structure can be readily calculated from (5.8) and (A8):

$$\frac{H_x}{H_0} = \begin{cases} \dfrac{2\pi|z|}{Q(\kappa)}, & \kappa > 1, \\ 0, & \kappa < 1, \end{cases} \qquad \frac{H_y}{H_0} = \begin{cases} \dfrac{2\pi\sigma}{Q(\kappa)}\left(\dfrac{2|A|}{u_c'}\right)^{1/2} k\sin\theta, & \kappa < 1, \\ 0, & \kappa > 1. \end{cases}$$

Hence it is evident that the maximum value of the x-component of the field tends to a constant value which is attained on the outer boundary of the cat's eye in its midsection (i.e. at points $|z| = 2 + 0,\ \theta = \pi$) and is

$$\left(H_x/H_0\right)_{max} = 4\pi/Q(1) = \pi/2\ .$$

The profile $H_x(Y)$ in the midsection of the cat's eye is shown in figure 11.

A maximum absolute value of the y-component of the field which is also attained on the outer boundary of the eye, at $\theta = \frac{1}{2}\pi, \frac{3}{2}\pi$, increases when eye is opening ($\sim |A|^{1/2}$), but even on the applicability boundary of the theory ($|A| \sim 1$) it remains of the order of the magnitude of an undisturbed field:

$$|H_y/H_0|_{max} = (\pi/4)k\left(2|A|/u_c'\right)^{1/2}\ .$$

Thus, a shear flow cannot transfer a substantial amount of energy to the magnetic field through an instability, and this kinematic dynamo mechanism is ineffective.

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Appendix A. Asymptotic behaviour of the function $\Phi_5(\lambda, P_m)$ in the limiting cases of a nonlinear ($\lambda \ll 1$) and viscous ($\lambda \gg 1$) CL for $P_m = O(1)$

We proceed from the system of equations (see § 5)

$$\mathscr{M}_\lambda g_5 = \frac{1}{4}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \theta}\left(f_1'\right)^2 - \left(f_1'\frac{\partial f_1}{\partial \theta}\right)'\right], \tag{A 1}$$

$$\mathscr{M}_{\lambda/P_m} f_1(\lambda/P_m; z, \theta) = 0, \quad \mathscr{M}_\mu = z\frac{\partial}{\partial \theta} + \sin\theta\frac{\partial}{\partial z} - \mu\frac{\partial^2}{\partial z^2} \tag{A 2}$$

with boundary conditions $g_5 \to 0$, $f_1 \to 2z$ as $z \to \pm\infty$, where the prime denotes the derivative with respect to z.

First we consider the limit $\lambda \to 0$, $\lambda/P_m \to 0$. It is convenient in this case to pass to the variables $\kappa = z^2/2 + \cos\theta$ and θ. Designating

$$h = \frac{\partial f_1}{\partial \kappa}, \quad G = g_5 + \frac{1}{4}\frac{\partial}{\partial \kappa}\left[h^2\left(1 - \cos\theta - z^2\right)\right], \tag{A 3}$$

and using the relationship

$$\frac{\partial h}{\partial \theta} = \frac{\lambda}{P_m}(zh)'', \tag{A 4}$$

resulting from (A 2), we rearrange (A 1) to give

$$\frac{\partial G}{\partial \theta} = \lambda\frac{\partial}{\partial \kappa}z\frac{\partial G}{\partial \kappa} - \frac{\lambda}{4P_m}\frac{\partial}{\partial \kappa}\left\{(zh)''h\left[z^2 - 2(1-\cos\theta)\right] + z\left[(zh)'\right]^2\right\}. \tag{A 5}$$

Here the prime now denotes the derivative with respect to κ, and $z = \sigma[2(\kappa - \cos\theta)]^{1/2}$, $\sigma = \mathrm{sign}(z)$. Next, we determine the order of the function G. To do so, we take advantage of the fact that f_1 is near the separatrix, i.e. in the region $|\kappa - 1| \sim O\left(\lambda^{1/2}\right)$, of $O\left(\lambda^{1/2}\right)$ (see Brown & Stewartson 1978, as well as the Appendix A of Churilov & Shukhman 1996). Consequently, h is in this same region, according to (A 3), of $O(1)$, and G is, according to (A 5), of $O(\lambda^{-1/2})$. Generally speaking, it should be expected that G has also the same order in the region outside the separatrix.

First we solve (A5) away from the separatrix, after that we perform a matching. Away from the separatrix we put

$$\left.\begin{aligned} G &= G^{(-1/2)}\lambda^{-1/2} + G^{(0)} + G^{(1/2)}\lambda^{1/2} + G^{(1)}\lambda + \cdots, \\ f_1 &= f_1^{(0)} + f_1^{(1)}\lambda + \cdots, \quad h = h^{(0)} + h^{(1)}\lambda + \cdots. \end{aligned}\right\} \tag{A 6}$$

It must be emphasized that the presence of the contribution $\sim O(\lambda^{-1/2})$ in this expansion for G is dictated solely by considerations of the matching through the cat's eye boundary.

The expansion (A 6) for f_1 (and hence for h) is well known (e.g. Haberman 1972).

For $f_1^{(0)}$ and $h^{(0)}$ we have

$$f_1^{(0)} = \begin{cases} 4\pi\sigma \displaystyle\int_1^{\kappa} \frac{\mathrm{d}x}{Q(x)}, & \kappa > 1, \\ 0, & \kappa < 1, \end{cases} \qquad h^{(0)} = \begin{cases} 4\pi\sigma/Q(\kappa), & \kappa > 1, \\ 0, & \kappa < 1, \end{cases} \tag{A 7}$$

where $Q(x) = \displaystyle\int_0^{2\pi} [2(\kappa - \cos\theta)]^{1/2} \mathrm{d}\theta$.

From (A 5) we have at $O(\lambda^{-1/2})$: $\partial G^{(-1/2)}/\partial\theta = 0$, whence

$$G^{(-1/2)} = U(\kappa). \tag{A 8}$$

The function $U(\kappa)$ is determined from the solvability condition of the equation for $G^{(1/2)}$

$$\frac{\partial G^{(1/2)}}{\partial\theta} = \frac{\partial}{\partial\kappa} z \frac{\partial U}{\partial\kappa}. \tag{A 9}$$

A standard procedure yields

$$U(\kappa) = \begin{cases} C_1 \displaystyle\int_1^{\kappa} \frac{\mathrm{d}x}{Q(x)} + C_2, & \kappa > 1, \\ C_3, & \kappa < 1. \end{cases}$$

It follows from (A 3) that away from the separatrix $G^{(-1/2)} = g_5^{(-1/2)}$. Therefore, taking into account that as $\kappa \to \infty$ in $g_5^{(-1/2)}$ there are no contributions proportional to $\kappa^{1/2}$ and const, one must put $C_1 = C_2 = 0$.

Thus, away from the separatrix we have

$$G^{(-1/2)} = U(\kappa) = \begin{cases} 0, & \kappa > 1, \\ C_3, & \kappa < 1. \end{cases} \tag{A 10}$$

From (A 10) and the definition of the function $\Phi_5(\lambda, P_m)$

$$\Phi_5(\lambda, P_m) = \fint_{-\infty}^{\infty} \langle g_5 \cos\theta \rangle \mathrm{d}z \tag{A 11}$$

it follows that when $\lambda \ll 1$

$$\Phi_5(\lambda, P_m) = -\frac{8C_3}{3\pi} \lambda^{-1/2} + \cdots, \tag{A 12}$$

and it is obvious that the entire contribution to the integral (A 11) at this order is made by the inner part of the cat's eye, $\kappa < 1$.

The problem is reduced to calculating C_3, i.e. to calculating the jump of the function $G^{(-1/2)}$ across the cat's eye boundary. To perform this calculation requires passing to the solution of equation (A5) near the separatrix, as done for the first time by Brown & Stewartson (1978) for the function f_1.

In this region we put

$$s = \lambda^{1/2}(\kappa - 1), \quad G = U(s,\theta)\lambda^{-1/2} + \cdots, \quad h = h^{(0)}(s,\theta) + \cdots$$

(we omit the upper index 0 of h in what follows). At the main order from (A 5) we obtain

$$\frac{\partial U}{\partial\theta} = z \frac{\partial^2 U}{\partial s^2} + \frac{1}{4} \left\{ \frac{1}{2} z^3 \left(h^2\right)''' - \frac{1}{P_m} z^3 \left[(h')^2\right]' \right\}, \tag{A 13}$$

where $z \equiv z(\theta) = 2\sigma \sin(\theta/2)$, the prime denoting the derivative with respect to s. We introduce the variable τ:

$$\tau = \begin{cases} 8\sin^2 \frac{1}{4}\theta, & \sigma > 0, \\ 8\cos^2 \frac{1}{4}\theta, & \sigma < 0, \end{cases}$$

where the angles take the value 0 and 8, and instead of (A 13) we get

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial s^2} + \frac{1}{4}\left\{\frac{1}{8}\tau(8-\tau)\left(h^2\right)''' - \frac{1}{4P_m}\tau(8-\tau)\left[(h')^2\right]'\right\}. \tag{A 14}$$

Since the function $G(z,\theta)$ is even with respect to the transformation $z \to -z$, $\theta \to 2\pi - \theta$, the periodicity condition for $U(s,\theta)$ may be written in the same form for $s > 0$ and $s < 0$:

$$U(s, \tau = 8) = U(s, \tau = 0)\ . \tag{A 15}$$

Boundary conditions for U follow from (A 10):

$$U(s,\tau) = \begin{cases} 0, & s \to +\infty, \\ C_3, & s \to -\infty. \end{cases} \tag{A 16}$$

Since in this region $f_1 = O(\lambda^{1/2})$ (see (A7)), we put $f_1 = u(s,\tau)\lambda^{1/2}$. Then

$$h(s,\tau) = \frac{\partial u(s,\tau)}{\partial s}, \tag{A 17}$$

where $u(s,\tau)$ is the solution of the equation $\partial u/\partial \tau = P_m^{-1}\partial^2 u/\partial s^2$ with boundary conditions

$$u = \begin{cases} \dfrac{\sigma\pi}{2}s, & s \to +\infty, \\ 0, & s \to -\infty, \end{cases} \tag{A 18}$$

satisfying the periodicity condition which, because of the oddness of $f_1(z,\theta)$, should be written as $u(s,0) = u(s,8)$ signs. For the function $u(s,\tau)$ we have (Brown & Stewartson 1978; Churilov & Shukhman 1996)

$$u(s,\tau) = \int_{-\infty}^{\infty} \mathrm{d}k\, u_k(0)\mathrm{e}^{\mathrm{i}ks - k^2\tau/P_m}, \tag{A 19}$$

where

$$u_k(0) = -\frac{\sigma}{\left[1 + \exp\left(-8k^2/P_m\right)\right] k^2 F_-(k/P_m^{1/2})}, \tag{A 20}$$

$$\ln F_\pm(k) = \frac{1}{2}\ln\frac{\tanh(4k^2)}{k^2} \pm iI(k), \quad F_\pm(0) = 2, \tag{A 21}$$

$$I(k) = \frac{8}{\pi}\int_0^\infty \frac{q\,dq}{\sinh(8q^2)}\ln\left|\frac{q-k}{q+k}\right| + \tfrac{1}{2}\pi\,\mathrm{sgn}\,k, \quad I(0) = 0. \tag{A 22}$$

The integration in (A 19) proceeds along the real axis with the indentation of the pole $k = 0$ from below. We now perform a Fourier-transform in (A 14):

$$\begin{aligned}\frac{\mathrm{d}U_k(\tau)}{\mathrm{d}\tau} + k^2 U_k(\tau) &= \frac{1}{4}\left\{\frac{1}{8}\,\tau(8-\tau)\left[\left(h^2\right)'''\right]_k - \frac{1}{4P_m}\,\tau(8-\tau)\left[\left(h'\right)^2\right]_k \mathrm{i}k\right\} \\ &\equiv R_k^{(1)}(\tau) + R_k^{(2)}(\tau) = R_k(\tau)\ ,\end{aligned} \tag{A 23}$$

where

$$R_k^{(1)} = \frac{1}{32}\,\tau(8-\tau)\left[\left(h^2\right)'''\right]_k, \quad R_k^{(2)} = -\frac{\mathrm{i}k}{16P_m}\,\tau(8-\tau)\left[\left(h'\right)^2\right]_k \ . \tag{A 24}$$

Next we solve (A 23):

$$U_k(\tau) = U_k(0)e^{-k^2\tau} + \int_0^\tau R_k(t)\mathrm{e}^{-k^2(\tau-t)}\mathrm{d}t \ .$$

Using the periodicity condition $U_k(8) = U_k(0)$, we find

$$U_k(0) = \frac{\displaystyle\int_0^8 R_k(t)\mathrm{e}^{-k^2(8-t)}\mathrm{d}t}{1-\exp(-8k^2)} + a\delta(k) + b\delta'(k) \ , \tag{A 25}$$

and, after performing an inverse Fourier-transform, we get

$$U(s,0) = a + bs + \int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}ks)\mathrm{d}k}{1-\exp(-8k^2)}\int_0^8 R_k(t)\,\mathrm{e}^{-k^2(8-t)}\mathrm{d}t \ . \tag{A 26}$$

Recall that in (A 26) the pole $k = 0$ is indented from below. When $s > 0$ the contour can be closed by a large-radius semicircle in the upper half-plane of the complex variable k. It is clear that the main contribution will be made by the pole $k = 0$. The other poles of the expression $\int_0^8 \mathrm{d}tR_k(t)\exp\left(-k^2(8-t)\right)$ and the zeros of the expression $\left(1-\exp(-8k^2)\right)$, lying in the upper half-plane, will make an exponentially small contribution when $s \to +\infty$. When $s < 0$ the contour may be closed in the lower half-plane, whence it follows that when $s \to -\infty$ the integral contains only exponentially small contributions from the poles of the integrand.

Since when $k \to 0$ the expression $\left(1-\exp(-8k^2)\right)$ has a second-order zero, the pole makes a finite contribution if the integral $\int_0^8 \mathrm{d}t\,R_k(t)\exp\left(-k^2(8-t)\right)$ at $k = 0$ is finite or has a first-order zero.

It can be shown that (see (A 24)) $J_k^{(1)} \equiv \int_0^8 \mathrm{d}tR_k^{(1)}(t)\,\mathrm{e}^{-k^2(8-t)} = O\left(k^2\right)$ as $k \to 0$ and does not make a finite contribution. Next, we examine the contribution from $R_k^{(2)}$:

$$J_k^{(2)} \equiv \int_0^8 \mathrm{d}t\,R_k^{(2)}(t)\,\mathrm{e}^{-k^2(8-t)} = -\frac{\mathrm{i}k}{16P_m}\int_0^8 \mathrm{d}t\,t(8-t)\,\mathrm{e}^{-k^2(8-t)}\left[\left(h'\right)^2\right]_k \ . \tag{A 27}$$

From (A 17), (A 19) we have

$$\left[\left(h'\right)^2\right]_k = \int \mathrm{d}k_1\,k_1^2(k-k_1)^2 u_{k_1}(0)u_{k-k_1}(0)\,\mathrm{e}^{-[k_1^2+(k-k_1)^2]\tau/P_m} \ . \tag{A 28}$$

It follows from (A 20), (A 21) that $[(h')^2]_k = O(1)$ as $k \to 0$. For $k \to 0$ we therefore write

$$J_k^{(2)} \approx -\frac{\mathrm{i}j}{4P_m}k, \quad j = \frac{1}{4}\int_0^8 \mathrm{d}t\,t(8-t)\left[\left(h'\right)^2\right]_{k=0} \ , \tag{A 29}$$

whence it is clearly seen that the integrand in (A 26) has at the point $k = 0$ the first-order pole. Using (A 20), (A 21) and the relationships $I(-k) = -I(k)$ and $F_-(-k) = F_+(k)$, we obtain

$$j = 6P_m^{1/2}C^{(5)}; \quad C^{(5)} = \frac{2}{3}\int_0^\infty \frac{\mathrm{d}z}{z^{3/2}}\left(\frac{1}{\mathrm{e}^z-1} - \frac{1}{z} + \frac{1}{2}\right) \ . \tag{A 30}$$

Using (A 29), (A 30) we obtain from (A 26), by taking the residues at the point $k = 0$:

$$U(s,0) = \begin{cases} a + bs + \frac{3}{8}\pi\, C^{(5)} P_m^{-1/2} + \cdots, & s \to +\infty\,, \\ a + bs + \cdots, & s \to -\infty\,, \end{cases} \tag{A 31}$$

where dots denote exponentially small contributions. Upon matching (A 31) to (A 16), we find that $b = 0$ and

$$a = C_3 = -\tfrac{3}{8}\pi\, C^{(5)} P_m^{-1/2}\,. \tag{A 32}$$

Upon substituting the resulting expression for C_3 in (A 12), we finally find for $\lambda \to 0$:

$$\Phi_5(\lambda, P_m) = C^{(5)} P_m^{-1/2} \lambda^{-1/2}\,. \tag{A 33}$$

The asymptotic representation for $\Phi_5(\lambda, P_m)$ in the limit of a viscous CL ($\lambda \to \infty$) is easy to reconstruct by comparing equation (5.13) with equation (4.5) written in real form. As a result, we obtain for $\lambda \to \infty$

$$\Phi_5(\lambda, P_m) = a_5(P_m)\lambda^{-7/3}\,. \tag{A 34}$$

Appendix B. Asymptotic behaviour of $\Phi_5(\lambda, P_m)$ for $P_m \ll 1$

When $P_m \ll 1$ it is necessary to distinguish three characteristic ranges of values of λ:

$$\text{(i)}\ \ \lambda \ll P_m, \quad \text{(ii)}\ \ P_m \ll \lambda \ll 1, \quad \text{(iii)}\ \ \lambda \gg 1.$$

In region (i) the CL is truly nonlinear, i.e. $\ell_N \gg \ell_{v_m} \gg \ell_v$. Obviously, in this region the representation of the function Φ_5 must coincide with that obtained in Appendix A for the case $\lambda \ll 1$ at $P_m = O(1)$. Therefore

$$\Phi_5(\lambda, P_m) = C^{(5)} P_m^{-1/2} \lambda^{-1/2}, \quad \lambda \ll P_m\,. \tag{B 1}$$

In region (iii) the CL is truly dissipative, i.e. $\ell_{v_m} \gg \ell_v \gg \ell_N$. Therefore, in this case the representation of Φ_5 also coincides with the representation obtained for $\lambda \gg 1$ at $P_m = O(1)$. We need only take advantage, in (A 34), of the asymptotic representation of $a_5(P_m)$ for $P_m \ll 1$ (see (4.6)): $a_5(P_m) = \frac{1}{2}|a_2|P_m$. Therefore

$$\Phi_5(\lambda, P_m) = \tfrac{1}{2}|a_2| P_m \lambda^{-7/3}, \quad \lambda \gg 1\,. \tag{B 2}$$

It remains to finish 'constructing' Φ_5 only in the intermediate region (ii). Here $\ell_v \ll \ell_N \ll \ell_{v_m}$. This means that the magnetic structure may be described in the approximation of the usual dissipative CL. We write equation (A2) as

$$z\frac{\partial g_1}{\partial \theta} - \frac{\lambda}{P_m}\frac{\partial^2 g_1}{\partial z^2} = -2\sin\theta - \sin\theta \frac{\partial g_1}{\partial z}\,, \tag{B 3}$$

where $g_1 = f_1 - 2z$. Next, we pass to the variable $x = z(\lambda/P_m)^{-1/3}$. From (B 3) we then obtain

$$x\frac{\partial g_1}{\partial \theta} - \frac{\partial^2 g_1}{\partial x^2} = -2(\lambda/P_m)^{-1/3}\sin\theta - (\lambda/P_m)^{-2/3}\sin\theta\frac{\partial g_1}{\partial x}\,. \tag{B 4}$$

When $\lambda/P_m \gg 1$ the second term on the right-hand side of equation (B 4) may be omitted, and it is easily solved thereafter:

$$g_1 = (\lambda/P_m)^{-1/3}\left[\Phi(x)\,e^{i\theta} + \overline{\Phi}(x)\,e^{-i\theta}\right],$$

where $\Phi(x) = \mathrm{i}\int_0^\infty \mathrm{d}t\,\exp(-t^3/3 - \mathrm{i}tx)$. Finally, for $f_1(\lambda/P_m, z, \theta)$ we find

$$f_1 = g_1 + 2z = 2z - 2(\lambda/P_m)^{-1/3}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^3/3}\sin\left(\theta - t\frac{z}{(\lambda/P_m)^{1/3}}\right). \qquad \text{(B 5)}$$

Substituting (B 5) in the right-hand side of (A 1) and linearizing it in f_1 gives

$$\mathscr{M}g_5 = \frac{1}{(\lambda/P_m)}\left\{\cos\theta + \frac{z}{(\lambda/P_m)^{1/3}}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t^3/3}\sin\left(\theta - t\frac{z}{(\lambda/P_m)^{1/3}}\right)\right\}. \qquad \text{(B 6)}$$

The function g_5 contains two contributions from the first and second terms on the right-hand side of (B6). It is easy to evaluate that the contribution to g_5 and Φ_5 from the second term is $(\lambda/P_m)^{2/3}$ times smaller than the contribution from the first term, and it may be neglected. From the resulting equation $\mathscr{M}_\lambda g_5 = (P_m/\lambda)\cos\theta$ it follows that in the region of parameters considered ($P_m \ll \lambda \ll 1$) the function $g_5(\lambda, P_m; z, \theta)$ may be expressed in terms of the function $g_2(\lambda; z, \theta)$ introduced earlier (e.g. Shukhman 1991; Churilov & Shukhman 1996) and satisfying the equation $\mathscr{M}g_2 = -2\cos\theta$:

$$g_5(\lambda, P_m; z, \theta) = -\tfrac{1}{2}(\lambda/P_m)^{-1}g_2(\lambda; z, \theta)\ .$$

Consequently $\Phi_5(\lambda, P_m) = -\frac{1}{2}(\lambda/P_m)^{-1}\Phi_2(\lambda)$ and finally we have

$$\Phi_5(\lambda, P_m) = \tfrac{1}{2}|C^{(2)}|P_m\lambda^{-2}, \quad P_m \ll \lambda \ll 1\ . \qquad \text{(B 7)}$$

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